2020-21 Onwards (MR-20)	MALLA REDDY ENGINEERING COLLEGE (Autonomous)	B.Tech. I Semester		
Code:A0B05	**Linear Algebra and Differential Equations**	**L**	**T**	**P**
Credits: 4	**(Common For CE, ME&MINING)**	**3**	**1**	**-**

Prerequisites: Matrices, Differentiation, and Integration

Course Objectives:

1. To learn rank of the matrix and its application to consistency of system of linear equations
2. To learn Eigen Values, Eigen Vectors and nature of Quadratic forms.
3. To learn the concept of the mean value theorems, partial differentiation and maxima and minima.
4. To learn methods of solving differential equations and its applications to basic engineering problems.
5. To learn basics of partial differential equations and the standard forms of partial differential equations.

Module -I: Matrix algebra [12 Periods]

Vector Space, basis, linear dependence and independence (Only Definitions)

Matrices: Types of Matrices, Symmetric; Hermitian; Skew-symmetric; Skew- Hermitian; orthogonal matrices; Unitary Matrices; rank of a matrix by Echelon form and Normal form, Inverse of Non-singular matrices by Gauss-Jordan method; solving system of Homogeneous and Non-Homogeneous linear equations. LU - Decomposition Method.

MODULE II: Eigen Values and Eigen Vectors [12 Periods]

Eigen values , Eigen vectors and their properties; Diagonalization of a matrix; Cayley-Hamilton Theorem (without proof); Finding inverse and power of a matrix by Cayley-Hamilton Theorem; Singular Value Decomposition.

Quadratic forms: Nature, rank, index and signature of the Quadratic Form, Linear Transformation and Orthogonal Transformation, Reduction of Quadratic form to canonical forms by Orthogonal Transformation Method.

Module - III: Differential Calculus [12 Periods]

Mean value theorems: Rolle’s theorem and Lagrange’s Mean value theorem with their Geometrical Interpretation and its applications, Cauchy’s Mean value Theorem. Taylor’s Series. Limits, Continuity, Partial differentiation, partial derivatives of first and second order, Jacobian, Taylor’s theorem of two variables (without proof). Maxima and Minima of two variables, Lagrange’s method of undetermined Multipliers.

Module –IV: Ordinary Differential Equations **[12 Periods]**
First Order and First Degree ODE: Orthogonal trajectories, Newton's law of cooling, Law of natural growth and decay.

Second and Higher Order ODE with Constant Coefficients: Introduction-Rules for finding complementary function and particular integral. Solution of Homogenous, non-homogeneous differential equations, Non-Homogeneous terms of the type e^{ax} , sin(ax), cos (ax), polynomials in x, e^{ax} V(x), x V(x), Method of variation of parameters.

Module – V: Partial Differential Equations **[12 Periods]**
Formation of partial differential equations by eliminating arbitrary constants or arbitrary function, solutions of first order linear (Lagrange) equations, solutions of non linear first order equations (four standard types). Equations reducible to linear, Charpit's Method.

Text Books:

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36th Edition, 2010.
2. R K Jain SRK Iyengar , Advanced engineering mathematics, Narosa publications.
3. Erwin Kreyszig, Advanced Engineering Mathematics, Wiley publications.

Reference Books:

1. G.B. Thomas and R.L. Finney, Calculus and Analytic geometry, 9th Edition, Pearson, Reprint,2002.
2. N.P. Bali and Manish Goyal, A text book of Engineering Mathematics, Laxmi Publications,
   Reprint, 2008.
3. V. Krishnamurthy, V.P. Mainra and J.L. Arora, An introduction to Linear Algebra, AffiliatedEast–West press, Reprint 2005.
4. Ramana B.V., Higher Engineering Mathematics, Tata McGraw Hill New Delhi, 11th Reprint,2010.

E – RESOURCES:

1.https://www.mathplanet.com/education/algebra-2/matrices/how-to-operate-with-matrices (Systems of linear equations, matrices)

2.http://math.mit.edu/~gs/linearalgebra/ila0601.pdf(Eigen values, Eigen vectors)

3http://www.math.cmu.edu/~wn0g/noll/2ch6a.pdf(Differential Calculus)

4. https://www.intmath.com/differential-equations/1-solving-des.php (Differential Equations)

5. https://www.math.uni-leipzig.de/~miersemann/pdebook.pdf (Partial differential Equations)

NPTEL:

1. https://www.youtube.com/watch?v=NEpvTe3pFIk&list=PLLy_2iUCG87BLKl8eISe4fHKdE2_j2B_T&index=5 (Matrices – System of linear Equations)
2. https://www.youtube.com/watch?v=wrSJ5re0TAw (Eigen values and Eigen vectors)
3. https://www.youtube.com/watch?v=yuE86XeGhEA (Quadratic forms)

Course Outcomes:

1. The student will be able to find rank of a matrix and analyze solutions of system of linear equations.
2. The student will be able to find Eigen values and Eigen vectors of a matrix, diagonalization a matrix, verification of Cayley Hamilton theorem and reduce a quadratic form into a canonical form through a linear transformation.
3. The student will be able to verify mean value theorems and maxima and minima of function of two variables.
4. Formulate and solve the problems of first and higher order differential equations
5. Apply knowledge of Partial differential equations in real world problems.

CO- PO Mapping:

CO- PO, PSO Mapping (3/2/1 indicates strength of correlation) 3-Strong, 2-Medium, 1-Weak												
COS	Programme Outcomes(POs)											
	PO 1	PO 2	PO 3	PO 4	PO 5	PO 6	PO 7	PO 8	PO 9	PO 10	PO 11	PO 12
CO1	3	2	2	3	3				2			3
CO2	3	2	2	3	2				2			3
CO3	3	2	2	3	2				2			2
CO4	3	2	2	3	3				2			2
CO5	3	2	2	3	3				2			2

MODULE -I

MATRIX ALGEBRA

MATRICES

Matrix :—

An arrangement of mn numbers (real or complex) in a rectangular array having m rows (Horizontal lines) and n columns (vertical lines), the numbers being enclosed by brackets [ ] or ( ) is called an m×n matrix (read as m by n matrix)

Here m×n is called as the order or type of a matrix and each of mn numbers is called as an element of matrix.

Generally matrices are denoted by capital letters A, B, C, ...

and its elements are denoted by small letters a, b, c, ...

An m×n matrix can be expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

It is briefly written as $A = [a_{ij}]_{m\times n}$

where $i = 1, 2, 3 \ldots m$ stands for rows

$j = 1, 2, 3 \ldots n$ stands for columns.

Eg :- $\begin{bmatrix} 7 & -2 & 0 \\ 8 & -3 & 1 \end{bmatrix}$ is a matrix of order 2×3

$\begin{bmatrix} 1 & 8 \\ 3 & 27 \end{bmatrix}$ is a matrix of order 2×2

Types of Matrices :—

Row Matrix :—

A matrix having only one row and any number of columns is said to be a row matrix. It is of order $1 \times n$.

Eg:- $\begin{bmatrix} -1 & 0 & 1 & 2 \end{bmatrix}$ is a row matrix of order 1×4

Column Matrix :—

A matrix having only one column and any number of rows is said to be a column matrix. It is of order $n \times 1$.

Eg:- $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is a column matrix of order 4×1

Rectangular Matrix :—

A matrix having rows and columns are not equal is said to be a rectangular matrix. It is of order $m \times n$.

Eg:- $\begin{bmatrix} 1 & 2 & 7 \\ 4 & 5 & 9 \end{bmatrix}$ is a rectangular matrix of order 2×3

Square Matrix :—

A matrix having rows and columns are equal is said to be a square matrix. It is of order $n \times n$ or square matrix of order n.

Eg:- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a square matrix of order 2.

Principal diagonal of a square matrix :—

In a matrix $A = [a_{ij}]_{n\times n}$, the diagonal which carries from the first row first element to last row last element is called the principal diagonal of A.

The elements a_{ij} of A for which $i=j$ i.e $a_{11}, a_{22}, a_{33}, \dots a_{nn}$ are called the elements of the principal diagonal of A.

Trace of a square matrix :—

Let $A = [a_{ij}]_{n\times n}$

The sum of the elements of the principal diagonal elements is called the Trace of A. and is denoted by tr A.

$$\therefore \quad tr A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \dots + a_{nn}$$

Properties :—

If A and B are square matrices of order n and λ is any scalar then

(i) $tr(\lambda A) = \lambda\, tr(A)$

(ii) $tr(A+B) = tr A + tr B$.

(iii) $tr(AB) = tr(BA)$.

Diagonal Matrix :—

A square matrix in which all the elements except in the principal diagonal are zero is called a diagonal matrix.

Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a square matrix of order 3.
is a diagonal matrix of order 3.

Eg:- If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$

Properties :- If A^T and B^T are the transposes of A and B respectively then

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$ Where A and B are of the same order.

(iii) $(KA)^T = KA^T$, Where K is a scalar

(iv) $(AB)^T = B^T A^T$ Where A and B are conformable for multiplication

Symmetric Matrix :-

A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j

Thus the necessary and sufficient condition for a square matrix A to be symmetric is that $A^T = A$

$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$ are symmetric matrices of order 3.

Skew Symmetric Matrix :-

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji}$ for every i and j

Thus A is skew symmetric matrix iff $A^T = -A$

Thus all the diagonal elements of a skew symmetric matrix are zero.

Eg: $\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$ are skew symmetric matrices of order 3.

Note :- (i) A is symmetric $\Longrightarrow$ KA is symmetric.

(ii) A is skew symmetric $\Longrightarrow$ KA is skew symmetric.

Properties :-

(i) Inverse of a non singular symmetric matrix A is symmetric.

(ii) If A is a symmetric matrix then adj A is also symmetric.

(iii) If A is a $m \times n$ matrix and B is a $n \times p$ matrix then $(AB)^T = B^T A^T$.

Theorem :- Every square matrix can be expressed as the sum of a symmetric and skew symmetric matrices in one and only way [OR] show that any square matrix $A = B + C$ where B is symmetric and C is skew symmetric matrices.

Proof :- Let A be any square matrix.

We can write $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$\because A = \frac{1}{2}A + \frac{1}{2}A$

$A = \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T + \frac{1}{2}A$

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$A = P + Q$ say.

where $P = \frac{1}{2}(A + A^T)$

$Q = \frac{1}{2}(A - A^T)$

We have $P = \frac{1}{2}(A + A^T)$

$$P^T = \left[\frac{1}{2}(A + A^T)\right]^T$$

$$= \frac{1}{2}(A + A^T)^T$$

$$= \frac{1}{2}\left(A^T + (A^T)^T\right)$$

$$P^T = \frac{1}{2}(A^T + A)$$

$$P^T = P.$$

$\therefore$ P is symmetric matrix.

We have $Q = \frac{1}{2}(A - A^T)$

$$Q^T = \left[\frac{1}{2}(A - A^T)\right]^T$$

$$= \frac{1}{2}(A - A^T)^T$$

$$Q^T = \frac{1}{2}(A - A^T)^T$$

$$= \frac{1}{2}\left(A^T - (A^T)^T\right)$$

$$= \frac{1}{2}(A^T - A)$$

$$= -\frac{1}{2}(A - A^T)$$

$$Q^T = -Q$$

$\therefore$ Q is skew symmetric matrix.

Thus, square matrix = symmetric + skew symmetric.

Thus, A is a sum of symmetric matrix and a skew symmetric matrix.

To Prove that the sum is unique :-

If possible, let $A = R + S$ be another such representation of A

Where R is a symmetric and S is a skew symmetric matrix.

$\therefore \quad R^T = R$ and $S^T = -S$.

Now $\quad A^T = (R+S)^T = R^T + S^T = R - S$

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}(R + S + R - S) = R.$$

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}(R + S - R + S) = S.$$

$\Rightarrow R = P$ and $S = Q$.

Thus, the representation is unique.

Express the matrix $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrices.

Sol:- Given that $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$

We know that symmetric part of matrix A is $P = \frac{1}{2}(A + A^T)$

and skew symmetric part of matrix A is $Q = \frac{1}{2}(A - A^T)$

$$A^T = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix} = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$$

$$P^T = P$$

$\therefore$ P is symmetric.

$$A - A^T = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

$$Q^T = -Q$$

$\therefore$ Q is skew symmetric.

$$A = P + Q = \frac{1}{2}\begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

Properties :-

(i) The inverse of a non singular symmetric matrix A is symmetric.

(ii) If A and B are symmetric matrices then AB is symmetric if and only if $AB = BA$.

(iii) If A is any matrix then AA^T and A^TA are both symmetric.

(iv) The matrix B^TAB is symmetric or skew symmetric according as A is symmetric or skew symmetric.

(v) All positive integral powers of a symmetric matrix are symmetric

(vi) Positive odd integral powers of a skew symmetric matrix are skew symmetric where as positive even integral powers are symmetric.

Orthogonal Matrix :—

A square matrix A is called an orthogonal matrix if $AA^T = A^TA = I$.

(1) Determine the values of a, b, c such that $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is an orthogonal matrix.

sol:- Given that $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$

By def. A is an orthogonal $\Rightarrow AA^T = A^TA = I$.

$$AA^T = I \Longrightarrow \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the elements of corresponding positions, we get

$4b^2+c^2 = 1$ — (1) $\quad 2b^2-c^2 = 0$ — (2)

$a^2+b^2+c^2 = 1$ — (3) $\quad a^2-b^2-c^2 = 0$ — (4)

$(1)+(2) \Longrightarrow 6b^2 = 1 \Longrightarrow b = \pm\frac{1}{\sqrt{6}}$

$(3)+(4) \Longrightarrow 2a^2 = 1 \Longrightarrow a = \pm\frac{1}{\sqrt{2}}$

from (2), $c^2 = 2b^2$

$c^2 = 2\cdot\frac{1}{6} = \frac{1}{3}$

$c = \pm\frac{1}{\sqrt{3}}$

$\therefore a = \pm\frac{1}{\sqrt{2}},\ b = \pm\frac{1}{\sqrt{6}}\ c = \frac{\pm 1}{\sqrt{3}}$ are required values.

(2) show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix}$ is an orthogonal.

(3) show that $A = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ is orthogonal.

(4) S/T. $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal.

(5) S/T $A = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi \end{bmatrix}$ is orthogonal.

(6) Find a +ve integer "a" such that $\frac{1}{a}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}$ is orthogonal.

Properties:-

(i) If A is orthogonal matrix then $|A| = \pm 1$

(ii) The inverse of an orthogonal matrix is orthogonal.

(iii) The transpose of an orthogonal matrix is orthogonal.

(iv) If A, B be orthogonal matrices, AB and BA are also orthogonal.

→ Reduce the matrix $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ to echelon form and find its rank.

Sol:- Let $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}_{4\times 4}$

Now we reduce the matrix A into echelon form by applying row operations only.

$R_2 \to R_2 + R_1 \quad R_3 \to R_3 + 2R_1 \quad R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

$R_3 \to 2R_3 - 11R_2, \quad R_4 \to R_4 + 2R_2$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_4 \to 6R_4 + R_3$

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is in echelon form

$\therefore \rho(A) =$ No. of non zero rows of the last equivalent to A $= 4$.

$\therefore \rho(A) = 4$

→ Apply elementary transformations to find the rank of

$$A = \begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}$$

Sol:- Given that $A = \begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}$

Now we reduce the matrix A into echelon form by applying row operations only.

$R_2 \to R_2 - 7R_1, \quad R_3 \to R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 69 & -23 & 46 \\ 0 & 33 & -11 & 22 \end{bmatrix}$$

$R_2 \to R_2\left(\frac{1}{23}\right) \quad R_3 \to R_3\left(\frac{1}{11}\right)$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 3 & -1 & 2 \\ 0 & 3 & -1 & 2 \end{bmatrix}.$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form,

$\therefore \rho(A)$ = The no. of non zero rows of the last equivalent of A $= 2$

$\therefore \rho(A) = 2.$

→ Find the constants λ and m such that the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & \lambda & m \end{bmatrix}$$ is (i) 3 (ii) 2

Sol:- Given that $$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & 2 & \lambda & m \end{bmatrix}$$

Now we reduce the matrix A into echelon form by applying row operations only.

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 6R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 10 & \lambda-18 & m-6 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & \lambda-4 & m-6 \end{bmatrix}$$

Which is in echelon form

(i) $\rho(A) = 3$ if $\lambda \neq 4$ or $m \neq 6$

(ii) $\rho(A) = 2$ if $\lambda = 4$ and $m = 6$.

→ For what value of k the matrix $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has rank 3.

Sol: Given that $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$

$R_2 \to 4R_2 - R_1,\ R_3 \to 4R_3 - kR_1,\ R_4 \to 4R_4 - 9R_1$

$$\sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$$

The given matrix is of order 4x4. If its rank is 3, then we must have $|A| = 0$

$$\Rightarrow 4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

$$1\left[(8-4k)3\right] - 1\left[(8-4k)(4k+27)\right] = 0.$$

$$(8-4k)\left[3 - 4k - 27\right] = 0$$

$$(8-4k)(-24-4k) = 0.$$

$\therefore\ k = 2$ or $k = -6$.

ECHELON FORM.

1 Define Echelon form of a matrix.

2 Find the rank of matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by reducing it into echelon form Ans:- 2.

3 Find the rank of matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ by reducing it into echelon form Ans:- 3

4 Find the value of K so that the rank of the matrix $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ K & 2 & 2 & 2 \\ 9 & 9 & K & 3 \end{bmatrix}$ is three. Ans:- $K = 2$ or $K = -6$.

5 Find the value of K, if the Rank of matrix A is 2 $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$

Ans:- $K = -2$

6 Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & 1 & 2 \\ -3 & -4 & 5 & 8 \\ 1 & 3 & 10 & 14 \end{bmatrix}$ by reducing it into echelon form Ans:- 2.

7 Find the rank of the matrix. $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ by reducing it into echelon form Ans:- 4.

8 Find the rank of the matrix $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ by reducing it into echelon form. Ans:- 2

Normal form or Canonical form of a matrix :-

If an $m \times n$ matrix can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by using a finite chain of elementary operations. Where I_r is the unit matrix of order r and '0' is the null matrix then the above form is called "The normal form" or "The first canonical form of a matrix". Here r indicates the rank of a matrix.

The various normal forms are I_r, $[I_r \; 0]$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ and $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Working procedure to reduce a matrix to the canonical form :-

Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Step (i) :- If $a_{11} \neq 0$, by using a_{11} position make a_{21} and a_{31} positions as zero. Here we use row operations.

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \end{bmatrix}$$

Step (ii) :- By using a_{11} position make a_{12}, a_{13} and a_{14} positions as zero. Here we use column operations.

$$\sim \begin{bmatrix} a_{11} & 0 & 0 & a \\ 0 & a''_{22} & a''_{23} & a''_{24} \\ 0 & a''_{32} & a''_{33} & a''_{34} \end{bmatrix}$$

Step (iii) :- If $a''_{22} \neq 0$, by using a''_{22} position make a''_{32} position as zero Here we use row operation.

$$\sim \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a''_{22} & a''_{23} & a''_{24} \\ 0 & 0 & a'''_{33} & a'''_{34} \end{bmatrix}$$

Step (iv) :- By using a''_{22} position make a''_{23} and a''_{24} positions as zero. Here we use column operations.

$$\sim \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a''_{22} & 0 & 0 \\ 0 & 0 & a^{iv}_{33} & a^{iv}_{34} \end{bmatrix}$$

Step (v) :- If $a^{iv}_{33} \neq 0$, by using a^{iv}_{33} position make a^{iv}_{34} position as zero. Here we use column operation.

$$\sim \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a''_{22} & 0 & 0 \\ 0 & 0 & a^{iv}_{33} & 0 \end{bmatrix}$$

Step (vi) :- By using suitable elementary operations make a_{11}, a''_{22} and a^{iv}_{33} positions as one. Now which is of the form

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$A \sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

$$\therefore \quad \rho(A) = 3.$$

→ Find the rank of a matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ by reducing it to canonical form.

Sol: Given that $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

Now we reduce the matrix A into normal form by applying row and column operations.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ $R_4 \to R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$ $C_3 \to C_3 - 3C_1$ $C_4 \to C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$ $R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - 2C_2$ $C_4 \to C_4 - 3C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2(-1)$

$$A \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is of the form $A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

which is in normal form.

$\therefore \rho(A) = 2$.

→ Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & K \end{bmatrix}$ by reducing it to the canonical form.

Sol:- Given that $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & K \end{bmatrix}$

Now we reduce the matrix A into ~~echelon~~ normal form by applying elementary row and column operations.

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 3R_1$ $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & K-3 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$ $C_3 \to C_3 + C_1$ $C_4 \to C_4 - 3C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & K-3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & K-3 \end{bmatrix}$$

$C_3 \to 7C_3 + 6C_2 \quad C_4 \to 7C_4 - 11C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -14 & 28 \\ 0 & 0 & 7 & 7K-21 \end{bmatrix}$$

$R_4 \to 2R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -14 & 28 \\ 0 & 0 & 0 & 14K-14 \end{bmatrix}$$

$C_4 \to C_4 + 2C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -14 & 0 \\ 0 & 0 & 0 & 14K-14 \end{bmatrix}$$

$R_3 \to R_3\left(-\frac{1}{7}\right) \quad R_4 \to R_4\left(-\frac{1}{14}\right)$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 14K-14 \end{bmatrix}$$

which is in normal form

$\rho(A) = 3$ if $14K-14=0$ i.e $K=1$

$\rho(A) = 4$ if $14K-14 \neq 0$ i.e $K \neq 1$

→ Reduce the matrix $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ to normal form and hence find the rank.

Sol. Let $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

Reduce the matrix A into normal form by applying row and column operations.

$C_1 \leftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$C_3 \to C_3 - 2C_1 \quad C_4 \to C_4 + 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 2C_3 - C_2, \quad C_4 \to 2C_4 - 3C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2\left(\frac{1}{4}\right)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Which is in normal form

$\therefore \rho(A) = 2$.

→ By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form, find its rank.

sol: Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

Reduce the matrix A into normal form by applying elementary row and column operations.

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -12 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1 \quad C_3 \to C_3 - 3C_1 \quad C_4 \to C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$C_3 \to 3C_3 - 2C_2,\ C_4 \to 3C_4 - 5C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -36 \end{bmatrix}$$

$R_2 \to R_2\left(-\frac{1}{3}\right) \quad R_3 \to R_3\left(-\frac{1}{36}\right)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

Which is in normal form

$\therefore \quad \rho(A) = 3.$

Elementary Matrix :-

It is a matrix obtained from a unit matrix by a single elemen-tary transformation.

Eg:- $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the elementary matrices obtained from I_3 by applying the elementary operations $E_1 \leftrightarrow C_3$, $R_1 \rightarrow R_1(3)$ and $R_1 \rightarrow R_1 + 3R_2$, respectively.

Theorem :-

Every elementary row (column) transformation of a matrix can be obtained by pre multiplication (post-multiplication) with corresponding elementary matrix.

Eg :- Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 9 \\ -2 & 5 & 6 \end{bmatrix}$

Let us interchange 1^{st} and 3^{rd} rows, we get $B = \begin{bmatrix} -2 & 5 & 6 \\ 2 & 3 & 9 \\ 1 & 3 & 5 \end{bmatrix}$

This B is same as the matrix obtained by pre multiplying A with the matrix E_{13} obtained from unit matrix by interchanging 1^{st} and 3^{rd} rows in it.

Verification :- $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$E_{13}A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 9 \\ -2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 6 \\ 2 & 3 & 9 \\ 1 & 3 & 5 \end{bmatrix}$$

Eg :-

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 9 \\ -2 & 5 & 6 \end{bmatrix}$$

Let us interchange 1^{st} and 3^{rd} columns, we get $B = \begin{bmatrix} 5 & 3 & 1 \\ 9 & 3 & 2 \\ 6 & 5 & -2 \end{bmatrix}$

This B is same as the matrix obtained by **post** multiplying A with the matrix E'_{13} obtained from unit matrix by inter-changing 1^{st} and 3^{rd} columns in it.

Verification :- $E'_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$AE'_{13} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 9 \\ -2 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 1 \\ 9 & 3 & 2 \\ 6 & 5 & -2 \end{bmatrix}$$

PAQ form of a Matrix :-

If A be an $m \times n$ matrix of rank-r, then there exists two non singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called PAQ form of a matrix A.

Working procedure :-

Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

We can write $A_{3\times3} = I_3 A I_3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we have to reduce the matrix A on the L.H.S to the normal form by applying elementary transformations. Each row transformation will be applied to the pre factor I_3 and each column transformation will be applied to the post factor I_3 on the R.H.S of equation ①.

Step (i) :- If $a_{11} \neq 0$, by using a_{11} position make a_{21} and a_{31} positions as zero. Here we apply row operations. The same row operations apply prefactor of A on R.H.S of ①.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step (ii) :- By using a_{11} position make a_{12} and a_{13} positions as zero. Here we apply column operations. The same column operations apply post factor of A.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & a''_{23} \\ 0 & a''_{32} & a''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Step(iii) :- If $a''_{22} \neq 0$, by using a''_{22} position make a''_{32} position as zero. Here we apply row operation. The same row operation apply on prefactor of A.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & a''_{23} \\ 0 & 0 & a'''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Step(iv) :- By using a''_{22} position make a''_{23} position as zero. Here we apply column operation. The same column operation apply on post factor of A.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & 0 \\ 0 & 0 & a^{iv}_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Step (v) :- By using elementary transformations reduce the matrix on L.H.S to an identity matrix. The same operations apply on prefactor or post factor on R.H.S.

The resultant is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

Where P and Q non singular matrices.

Note :- Here the non singular matrices P and Q are not unique.

→ Obtain the non singular matrices P and Q such that PAQ is in the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ Also find the rank of the matrix A.

Sol: Given that $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}_{3\times3}$.

We can write $A = I_3 A I_3$ —— ①

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we have to reduce the matrix A on the L.H.S to the normal form by applying elementary transformations.

Each elementary row transformation will be applied to the prefactor I_3 and each elementary column transformation will be applied to the post factor I_3 of the R.H.S of equation ①.

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1 \quad C_3 \to C_3 - C_1$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1 \quad C_3 \to C_3 - C_1 \quad C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 5R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -1 & -2 \\ 0 & 0 & -24 & -48 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 5C_3 - C_2 \quad , \quad C_4 \to 5C_4 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -120 & -240 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & -8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$C_4 \to C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -120 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & -10 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$R_2 \to R_2\left(-\frac{1}{5}\right), \quad R_3 \to R_3\left(-\frac{1}{120}\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/5 & 2/5 \\ \frac{-1}{24} & \frac{1}{120} & \frac{1}{15} \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & -10 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

This is of the form $[I_3 \;\; 0] = PAQ$

Where $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/5 & 2/5 \\ \frac{-1}{24} & \frac{1}{120} & \frac{1}{15} \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -1 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & -10 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Here P and Q are non singular matrices.

$\therefore \; \rho(A) = 3$.

PAQ Form of a Matrix

1 Find the matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$ Ans:- 2.

2 Find the matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$ Ans:- 2.

3 Find the non singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$$ Ans:- 2

4 Find the non singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$ Ans:- 3.

5 Find the non singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 4 & -3 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$ Ans:- 3

6 Find the non singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$$ Ans:- 3.

The Inverse of a Non Singular Matrix by Elementary Transformations:-

(Gauss Jordan Method) :-

We can find the inverse of a non singular matrix by using elementary row operations only. This method is known as Gauss Jordan Method.

If a non singular matrix A of order n is reduced to the unit matrix I_n by sequence of E-row transformations only, then the same sequence of E-row transformations applied to the unit matrix I_n gives the inverse of A i.e A^{-1}.

Working Procedure to find inverse of non singular matrix by using row operations :-

Suppose A is a non singular matrix of order 3.

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{We write} \quad A_{3\times3} = I_3 A \quad \text{—— (1)}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Now we reduce the matrix A on the L.H.S to be identity matrix I_3 by applying E-row transformations only. Each E-row transformation will be applied to the prefactor I_3 of the R.H.S of eqn (1).

Step(i) :- If $a_{11} \neq 0$, by using a_{11} position make a_{21} and a_{31} positions as zero. Here we apply row operations. The same operations apply on prefactor of A on R.H.S of (1)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A$$

Step (ii) :- If $a'_{22} \neq 0$, by using a'_{22} position make a_{12} and a'_{32} positions as zero. Here we apply row operations. The same operations apply on prefactor of A.

$$\begin{bmatrix} a'_{11} & 0 & a'_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A$$

Step (iii) :- If $a''_{33} \neq 0$, by using a''_{33} position make a'_{23} and a'_{13} positions as zero. Here we apply row operations. The same operations apply on prefactor of A.

$$\begin{bmatrix} a''_{11} & 0 & a \\ 0 & a''_{22} & 0 \\ 0 & 0 & a''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A$$

Step (iv) :- $R_1 \longrightarrow R_1\left(\frac{1}{a''_{11}}\right)$, $R_2 \longrightarrow R_2\left(\frac{1}{a''_{22}}\right)$, $R_3 \longrightarrow R_3\left(\frac{1}{a''_{33}}\right)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = BA$$

$$I = BA$$

$\therefore$ B is called inverse of A.

→ Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ by using elementary transformations.. verify $AA^{-1} = I$

Sol:- Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

We can write $A = I_3 \cdot A$ —— ①

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Now reduce the matrix A on L.H.S to the Identity matrix I_3 by using E-row transformations only. Each row transformation will be applied to the prefactor I_3 of the R.H.S of equation ①.

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1 \quad R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} A$$

$R_1 \to 3R_1 + R_2 \quad R_3 \to 3R_3 - 2R_2$

$$\begin{bmatrix} 3 & 0 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ -2 & 1 & 3 \end{bmatrix} A$$

$R_1 \to R_1 + 2R_3 \quad R_2 \to 2R_2 + R_3$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 6 \\ 0 & -3 & 3 \\ -2 & 1 & 3 \end{bmatrix} A$$

$R_1 \to R_1\left(\frac{1}{3}\right)$ $R_2 \to R_2\left(\frac{-1}{6}\right)$ $R_3 \to R_3\left(\frac{-1}{2}\right)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

which is of the form $I_3 = BA$

Here B is called Inverse of A. [$\because$ By def.]

$$\therefore \quad B = A^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

Verification :-

$$AA^{-1} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore AA^{-1} = I$.

Working Procedure to find Inverse of non singular matrix by using column operations :-

Suppose A is a non singular matrix of order 3.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We write $A_{3\times3} = A I_3$ ——— ①

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we reduce the matrix A on the L.H.S to be identity matrix I_3 by applying E-column transformations only. Each E-column transformation will be applied to the post factor I_3 of the R.H.S of eqn ①.

Step (i) :- If $a_{11} \neq 0$, by using a_{11} position make a_{12} and a_{13} positions as zero. Here we apply column operations. The same operations apply on post factor of A on R.H.S of ①.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} = A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

step (ii) :- If $a_{22}' \neq 0$ by using a_{22}' position make a_{21} and a_{23}' positions as zero. Here we apply column operations. The same operations apply on post factor of A

$$\begin{bmatrix} a_{11}' & 0 & 0 \\ 0 & a_{22}' & 0 \\ a_{31}' & a_{32}' & a_{33}'' \end{bmatrix} = A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

step(iii) :- If $a''_{33} \neq 0$, by using a''_{33} position make a'_{31}, a'_{32} positions as zero. Here we apply column operations. The same operations apply on ~~post~~factor of A

$$\begin{bmatrix} a''_{11} & 0 & 0 \\ 0 & a''_{22} & 0 \\ 0 & 0 & a''_{33} \end{bmatrix} = A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

step(iv) :- $C_1 \rightarrow C_1\left(\frac{1}{a''_{11}}\right)$ $C_2 \rightarrow C_2\left(\frac{1}{a''_{22}}\right)$ $C_3 \rightarrow C_3\left(\frac{1}{a'_{33}}\right)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AB$$

$$I = BA$$

$\therefore$ B is called inverse of A.

Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ by using elementary column transformations.

verify $AA^{-1} = I$.

sol:- Given that $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

We can write $A = AI_3$

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we reduce the matrix A on L.H.S to the Identity matrix I_3 by using E-column transformations only. Each column transformation will be applied to the post factor I_3 of the R.H.S of equation ①.

$C_2 \to 2C_2 + C_1 \quad C_3 \to 2C_3 - 3C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$C_1 \to 3C_1 - C_2 \;, \quad C_3 \to 3C_3 + C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 4 & -1 & -4 \end{bmatrix} = A \begin{bmatrix} 2 & 1 & -8 \\ -2 & 2 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

$C_1 \to C_1 + C_3 \quad C_2 \to 4C_2 - C_3$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -4 \end{bmatrix} = A \begin{bmatrix} -6 & 12 & -8 \\ 0 & 6 & 2 \\ 6 & -6 & 6 \end{bmatrix}$$

$C_1 \to C_1\left(\frac{1}{6}\right), \quad C_2 \to C_2\left(\frac{1}{12}\right), \quad C_3 \to C_3\left(-\frac{1}{4}\right),$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

which is of the form $I_3 = AB$.

Here B is called inverse of A. [$\because$ By def]

$$\therefore \quad B = A^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

Verification :-

$$AA^{-1} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad AA^{-1} = I.$$

INVERSE OF MATRIX

1. Define Inverse of matrix

2. Employing elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ Ans: $A^{-1} = \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$

3. Employing elementary column transformations, find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$ Ans: $A^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$

4. Find the inverse of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ by elementary column transformations. Ans: $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

5. Employing elementary row transformations find the inverse of the matrix $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ Ans: $A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 6 \end{bmatrix}$

6. Find the inverse of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ by using elementary row transformations.

Ans:- $A^{-1} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$

7. Find the inverse of matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ by using elementary column transformations.

Ans:- $\begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}$

System of simultaneous linear non-homogeneous equations :-

A system of m simultaneous non homogeneous linear equations in n unknowns $x_1, x_2, x_3, \dots x_n$ is of the form.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ \dots \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m \end{array}\right\} \text{——} (1)$$

We can write the above system of equations (1) in the form of matrix equation given by $AX = B$ —— (2)

i.e $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m\times n}$ is called the coefficient matrix of the system of equations (1)

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ is the matrix of unknowns and.

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$ be the constant matrix of the system of equations (1)

The set of values $x_1, x_2, x_3, \dots x_n$ which satisfy the system (1) is called the solution of the system.

Augmented Matrix :—

The matrix $[A|B] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right]$ is said to be the

augmented matrix of the given system of non homogeneous equations.

Consistency and In consistency :—

Any system of equations which contains one or more solutions is said to be consistent otherwise it is said to be inconsistent i.e the inconsistent system does not contain any solution.

Condition for consistency (Rank Test) :—

The necessary and sufficient condition for a system of non homogeneous equations $AX = B$ is said to be consistent is that the rank of the co efficient matrix A is same as the rank of the augmented matrix $[A|B]$. Then the system of equations $AX = B$ is consistent

$\iff \rho(A) = \rho([A|B])$

Note :- If $\rho(A) \neq \rho([A|B])$ then the given system $AX = B$ is inconsistent

Working Procedure :—

Suppose we have m equations in n unknowns.

The matrix equation of the given system of equations is $AX = B$. Then the co efficient matrix A is of order $m \times n$. Now write the augmented matrix $[A|B]$.

Step 1 :- First reduce the augmented matrix $[A|B]$ to echolon form by applying E-row operations only. With this we get the ranks of the augmented matrix $[A|B]$ and the co efficient matrix A.

Step 2 :-

Case (i) :- When $\rho(A) \neq \rho([A|B])$

In this case the given system of equations i.e $AX = B$ is inconsistent i.e it has no solution.

Case (ii) :- When $\rho(A) = \rho([A|B]) = r$ say

In this case the given system of equations i.e $AX=B$ is consistent i.e it contains a solution.

Now we have to verify the following points.

(a) If $r=n$ i.e the no. of unknowns then the given system has a unique solution.

(b) If $r<n$ i.e the no. of unknowns, the given system contains an infinite no. of solutions. To determine these solutions we have to assign an arbitrary value for $(n-r)$ variables and the remaining are depending upon them.

(c). If $m<n$ i.e the no. of equations less than the no. of unknowns then since $r \le m < n$, the given system possesses an infinitinite no. of solutions.

Properties :-

Symmetric and skew symmetric matrices :—

Theorem: A necessary and sufficient condition for a matrix A to be symmetric is that $A^T = A$. (OR) A is symmetric $\Leftrightarrow A^T = A$.

Proof :- A is symmetric $\Rightarrow A^T = A$.

Let $A = [a_{ij}]$ be an n- rowed square matrix so that $a_{ij} = a_{ji}$

Also A^T is also an n- rowed square matrix and

the $(i,j)^{th}$ element of A^T = the $(j,i)^{th}$ element of A.

$= a_{ji}$

$= a_{ij} = (i,j)^{th}$ element of A.

Hence $A^T = A$.

Converse: $A^T = A \Rightarrow$ A is symmetric.

Now $(i,j)^{th}$ element of A = $(i,j)^{th}$ element of A^T (Given $A^T = A$)

$= (j,i)^{th}$ element of A.

Hence A is symmetric matrix.

Theorem :- A necessary and sufficient condition for a matrix A to be skew symmetric matrix is that $A^T = -A$ (OR) A is skew symmetric $\Leftrightarrow A^T = -A$

Proof :- A is skew symmetric $\Rightarrow A^T = -A$.

Let A be an n- rowed skew symmetric matrix.

So that $a_{ij} = -a_{ji}$

Now A^T is also n- rowed square matrix $(i,j)^{th}$ element of A^T

$= (j,i)^{th}$ element of A.

$a_{ji} = -a_{ij}$ = the $(i,j)^{th}$ element of $-A$.

Hence $A^T = -A$.

Converse : $A^T = -A \Rightarrow A$ is skew symmetric

Now $(i,j)^{th}$ element of A = the negative of $(i,j)^{th}$ element of A^T

= the negative of $(j,i)^{th}$ element of A

A is a skew symmetric matrix.

Theorem :- The inverse of a non singular matrix A is symmetric.

Proof :- A is non singular matrix $\Rightarrow A^{-1}$ exists $\Rightarrow A^T = A$.

Now $(A^{-1})^T = (A^T)^{-1} = A^{-1}$

$(A^{-1})^T = A^{-1} \Rightarrow A^{-1}$ is symmetric.

Theorem :- If A and B are symmetric matrices then AB is symmetric if and only if $AB = BA$.

Proof :- Given A and B are symmetric $\Rightarrow A^T = A$ and $B^T = B$.

Suppose $AB = BA$

Consider $(AB)^T = B^T A^T = BA = AB$.

$(AB)^T = AB \Rightarrow AB$ is symmetric.

Conversely, suppose AB is symmetric.

$\Rightarrow AB = (AB)^T = B^T A^T = BA$.

$\Rightarrow AB = BA$.

Hence AB is symmetric if and only if $AB = BA$.

Theorem :- If A be any matrix then AA^T and A^TA are both symmetric matrices

Proof :- Let A be any matrix.

Now $(AA^T)^T = (A^T)^T A^T = AA^T \Rightarrow AA^T$ is symmetric.

Also $(A^TA)^T = A^T (A^T)^T = A^TA \Rightarrow A^TA$ is symmetric.

Theorem:- The matrix B^TAB is symmetric or skew symmetric according as A is symmetric or skew symmetric.

Proof:- (i) Let A be symmetric matrix $\Rightarrow A^T = A$.

Now $(B^TAB)^T = B^TA^T(B^T)^T = B^TAB$

$\Rightarrow B^TAB$ is symmetric.

(ii) Let A be skew symmetric matrix $\Rightarrow A^T = -A$.

Now $(B^TAB)^T = B^TA^T(B^T)^T = B^T(-A)B = -B^TAB$.

$\therefore (B^TAB)^T = -B^TAB$

$\Rightarrow B^TAB$ is skew symmetric.

Theorem:- All positive integral powers of a symmetric matrix are symmetric.

Proof:- Let A be symmetric matrix.

Now $A^n = A.A.A \dots A$ upto n times where n is a +ve integer.

$(A^n)^T = (A.A.A \dots A \text{ upto } n \text{ times})^T$

$= A^TA^TA^T \dots A^T$ upto n times.

$= AAA \dots A$ upto n times.

$= A^n$

$\therefore (A^n)^T = A^n \Rightarrow A^n$ is symmetric.

Theorem:- Positive odd integral powers of a skew symmetric matrix are skew symmetric where as positive even integral powers are symmetric.

Proof: Let A be a skew symmetric matrix $\Rightarrow A^T = -A$

Now $(A^n)^T = (A.A.A \dots A \text{ n times})^T = A^TA^TA^T \dots A^T$ n times

$= (-A)(-A)(-A) \dots (-A)$ n times.

$= (-1)^n A^n$ where n is a +ve integer

$= -A^n$ or A^n according as n is odd or even.

If n is an odd +ve integer, then $(A^n)^T = -A^n \Rightarrow A^n$ is skew symmetric.

If n is an even +ve integer then $(A^n)^T = A^n \Rightarrow A^n$ is symmetric.

Properties of orthogonal matrix :—

Theorem :- If A is orthogonal matrix, then $|A| = \pm 1$

Proof :- Given A is orthogonal matrix $\Rightarrow A^T A = I$.

$$\Rightarrow |A^T A| = |I|.$$

$$\Rightarrow |A^T||A| = 1.$$

$$|A||A| = 1 \qquad [\because |A^T| = |A|].$$

$$|A|^2 = 1$$

$$|A| = \pm 1.$$

Since $|A| \neq 0$, A is invertible.

Now $A^T A = I \Rightarrow A^T (A A^{-1}) = I A^{-1}$

$$A^T I = A^{-1}$$

$$A^T = A^{-1}.$$

Note :- A is orthogonal $\Rightarrow A A^T = I = A^T A$.

A is orthogonal $\Rightarrow A^T = A^{-1}$

Theorem : If A, B be orthogonal matrices. AB and BA are also orthogonal.

Proof :- Let A and B are n-rowed square matrices.

$|AB| = |A||B| \Rightarrow |AB| \neq 0$ since $|A| \neq 0$ and $|B| \neq 0$.

$$(AB)^T = B^T A^T$$

$$(AB)^T (AB) = (B^T A^T)(AB)$$

$$= B^T (A^T A) B$$

$$= B^T I B$$

$$= B^T B$$

$$= I.$$

[$\because$ A is orthogonal

$AA^T = A^T A = I$

B is orthogonal

$BB^T = B^T B = I$

$$(AB)^T (AB) = I$$

$\Rightarrow AB$ is orthogonal.

Similarly $(AB)(AB)^T = I$

$\Rightarrow AB$ is orthogonal.

$$(BA)(BA)^T = (BA)(A^T B^T)$$
$$= B(AA^T)B^T$$
$$= BIB^T$$
$$= BB^T$$
$$(BA)(BA)^T = I$$
$\Rightarrow BA$ is orthogonal

Similarly $(BA)^T(BA) = (A^T B^T)(BA)$
$$= A^T(B^T B)A$$
$$= A^T I A$$
$$= A^T A$$
$$(BA)^T(BA) = I$$
$\Rightarrow BA$ is orthogonal.

Verify that the determinant of an orthogonal matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is ± 1

sol: Given that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$|A| = \cos^2\theta + \sin^2\theta = 1$$

$$|A| = 1.$$

If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ are orthogonal matrices

then prove that AB and BA are orthogonal.

sol:- Given that $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

$$AB = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$AB = \frac{1}{3}\begin{bmatrix} \cos\theta - 2\sin\theta & 2 & \sin\theta + 2\cos\theta \\ 2\cos\theta + 2\sin\theta & 1 & 2\sin\theta - 2\cos\theta \\ -2\cos\theta + \sin\theta & 2 & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(AB)^T = \frac{1}{3}\begin{bmatrix} \cos\theta - 2\sin\theta & 2\cos\theta + 2\sin\theta & -2\cos\theta + \sin\theta \\ 2 & 1 & 2 \\ \sin\theta + 2\cos\theta & 2\sin\theta - 2\cos\theta & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(AB)(AB)^T = \frac{1}{9}\begin{bmatrix} \cos\theta - 2\sin\theta & 2 & \sin\theta + 2\cos\theta \\ 2\cos\theta + 2\sin\theta & 1 & 2\sin\theta - 2\cos\theta \\ -2\cos\theta + \sin\theta & 1 & -2\sin\theta - \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta - 2\sin\theta & 2\cos\theta + 2\sin\theta & -2\cos\theta + \sin\theta \\ 2 & 1 & 2 \\ \sin\theta + 2\cos\theta & 2\sin\theta - 2\cos\theta & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(AB)(AB)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$ $\therefore$ AB is an orthogonal matrix.

$$BA = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$BA = \frac{1}{3}\begin{bmatrix} \cos\theta - 2\sin\theta & 2\cos\theta + 2\sin\theta & 2\cos\theta - \sin\theta \\ 2 & 1 & -2 \\ -\sin\theta - 2\cos\theta & -2\sin\theta + 2\cos\theta & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(BA)^T = \frac{1}{3}\begin{bmatrix} \cos\theta - 2\sin\theta & 2 & -\sin\theta - 2\cos\theta \\ 2\cos\theta + 2\sin\theta & 1 & -2\sin\theta + 2\cos\theta \\ 2\cos\theta - \sin\theta & -2 & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(BA)(BA)^T = \frac{1}{9}\begin{bmatrix} \cos\theta - 2\sin\theta & 2\cos\theta + 2\sin\theta & 2\cos\theta - \sin\theta \\ 2 & 1 & -2 \\ -\sin\theta - 2\cos\theta & -2\sin\theta + \cos\theta & -2\sin\theta - \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta - 2\sin\theta & 2 & -\sin\theta - 2\cos\theta \\ 2\cos\theta + 2\sin\theta & 1 & -2\sin\theta + \cos\theta \\ 2\cos\theta - \sin\theta & -2 & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(BA)(BA)^T = I.$$

BA is an orthogonal matrix.

Theorem:- The inverse of an orthogonal matrix is orthogonal.

proof:- Let A be an orthogonal matrix $\Rightarrow AA^T = I = A^TA$

Taking inverse $\Rightarrow (AA^T)^{-1} = I^{-1} = (A^TA)^{-1}$

$$(A^T)^{-1}A^{-1} = I = A^{-1}(A^T)^{-1}$$

$$(A^{-1})^T A^{-1} = I = (A^{-1})(A^{-1})^T$$

$\Rightarrow A^{-1}$ is an orthogonal

Theorem:- The transpose of an orthogonal matrix is orthogonal.

proof:- Let A be an orthogonal matrix $\Rightarrow AA^T = I = A^TA$.

Taking transpose $\Rightarrow (AA^T)^T = I^T = (A^TA)^T$

$$(A^T)^T A^T = I = A^T(A^T)^T$$

$\Rightarrow A^T$ is orthogonal.

Eg:- Proove that inverse of an orthogonal matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol Given that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$|A| = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

$$A^{-1} = \frac{1}{|A|} \text{Adj}\, A$$

$$A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$(A^{-1})(A^{-1})^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$(A^{-1})(A^{-1})^T = I.$$

$\therefore A^{-1}$ is an orthogonal.

Eg:- Prove that transpose of an orthognal matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol:- Given that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$(A^T)(A^T)^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$(A^T)(A^T)^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ A^T is an orthogonal.

Idempotent Matrix :—

A square matrix A is said to be Idempotent if $A^2 = A$.

Eg:- $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$

$$A^2 = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} = A.$$

$$A^2 = A$$

$\therefore$ A is an idempotent matrix.

1) show that the matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Nilpotent Matrix :—

If A is a square matrix such that $A^m = 0$ where m is a least positive integer then A is called nilpotent.

If m is least +ve integer such that $A^m = 0$ then A is called nilpotent of index m.

Eg:- $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 = 0$.

$\therefore$ A is a nilpotent matrix of index 3.

(1) s/T for any real values of a and b the matrix $\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is nilpotent of index 2.

(2) s/T for $a \neq 0, b \neq 0$ the matrix $\begin{bmatrix} a & -b & -(a+b) \\ -a & b & a+b \\ a & -b & -(a+b) \end{bmatrix}$ is a nilpotent matrix index 2.

Involutary Matrix :-

If A is a square matrix such that $A^2 = I$ (I is unit matrix of order same as that of A) then A is said to be Involutary.

Eg :- $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A^2 = I$

$\Longrightarrow$ A is involutary.

(1) S/T $A = \begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix}$ is involutary.

(2) S/T $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ is involutary.

Periodic Matrices :—

If A is a square matrix such that $A^{n+1} = A$ where n is a +ve integer then A is called a periodic matrix.

If n is the least +ve integer satisfying the relation $A^{n+1} = A$ then n is called the period of A.

Eg:- $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$A^2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^2 = A$$

$\therefore$ A is periodic of order one.

(1) PIT $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a periodic matrix and its period is 4.

DETERMINANTS :—

Determinant of a 2×2 matrix :—

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix of order 2, then the value $ad - bc$ is called the determinant of A. It is denoted by det A or |A|

i.e $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Eg :- If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $|A| = 4 - 6 = -2$

Minors and Cofactors of a square matrix :—

(a) The minor of an element a_{ij} in a determinant is obtained by omitting the row and the column of the a_{ij}. It is denoted by M_{ij}.

(b) The cofactor of an element a_{ij} in a determinant is obtained by multiplying its minor with $(-1)^{i+j}$, where i, j indicate the row and column of the element a_{ij}. It is denoted by A_{ij}.

i.e $A_{ij} = (-1)^{i+j} M_{ij} \quad \forall i, j$

Eg :- If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 7 \\ -6 & 5 & 8 \end{bmatrix}$

(i) The minor of an element 4 is

$$M_{22} = \begin{vmatrix} 1 & 3 \\ -6 & 8 \end{vmatrix} = 8 + 18 = 26.$$

(ii) The cofactor of an element 4 is

$$A_{22} = (-1)^{2+2} M_{22} = 26.$$

Determinant of an nxn matrix :—

The sum of the products of the elements of any row or any column by its corresponding cofactors is said to be the determinant of a matrix of order n.

We can expand the determinant in terms of any row or any column of the matrix.

Thus if $A = [a_{ij}]_{n\times n}$ then

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \cdots + a_{1n}A_{1n}$$
$$= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + \cdots + a_{2n}A_{2n}$$
$$\cdots$$
$$= a_{n1}A_{n1} + a_{n2}A_{n2} + a_{n3}A_{n3} + \cdots + a_{nn}A_{nn}.$$
$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + \cdots + a_{n1}A_{n1}.$$
$$\cdots$$
$$= a_{1n}A_{1n} + a_{2n}A_{2n} + a_{3n}A_{3n} + \cdots + a_{nn}A_{nn}.$$

Thus if $A = [a_{ij}]_{3\times 3}$ then

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$
$$= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$
$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$
$$\cdots$$

$$|A| = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

Where $A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$$\therefore |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Find the determinant of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol: Given that $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$|A| = 8 \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} - (-6) \begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$|A| = 40 - 60 + 20$$

$$|A| = 0.$$

Note :— (i) If A is a square matrix of order n and k is any scalar then $|kA| = k^n|A|$.

(ii) If A is a square matrix of order n, Then $|A| = |A^T|$.

(iii) If A and B be two square matrices of same order Then $|AB| = |A||B|$.

Eg : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ Then

$$|A| = a_{11}.A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 1(-2-4) + 2(4-12) + 3(2+3)$$

$$|A| = -6 + 16 + 15 = 25.$$

Adjoint of a Matrix :—

If A is a square matrix of order n, then the transpose of the cofactor matrix of A is said to be the adjoint of a matrix A. It is denoted by adj A.

Thus if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the cofactor matrix of

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$\therefore$ Adj A = [The cofactor matrix of A]^T = $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Note :- If A is a square matrix of order n, then $A(\text{adj}A) = (\text{adj}A).A = |A|.I$ where I is a unit matrix of order n.

Eg:- $A = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -3 & -1 \\ -4 & 1 & 3 \end{bmatrix}$

Cofactor of an element $a_{23} = -1$ is $A_{23} = (-1)^{2+3} \begin{vmatrix} 6 & 2 \\ -4 & 1 \end{vmatrix}$

$$A_{23} = -(6+8)$$

$$A_{23} = -14$$

Cofactor of an element $a_{31} = -4$ is $A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix}$

$$A_{31} = -2+12$$

$$A_{31} = 10$$

Cofactor of an element $a_{22} = -3$ is $A_{22} = (-1)^{2+2} \begin{vmatrix} 6 & 4 \\ -4 & 3 \end{vmatrix}$

$$A_{22} = 18+16$$

$$A_{22} = 34$$

Cofactor of an element $a_{12} = 2$ is $A_{12} = (-1)^{1+2} \begin{vmatrix} -2 & -1 \\ -4 & 3 \end{vmatrix}$

$$A_{12} = -(-6-4)$$

$$A_{12} = 10$$

Inverse of a Matrix :-

Let A be any square matrix then a matrix B if exists such that $AB = BA = I$ then B is called inverse of A and is denoted by A^{-1}.

Singular matrix :- A square matrix A is said to be singular if $|A| = 0$.

Non singular matrix :- A square matrix A is said to be non singular if $|A| \neq 0$.

→ Thus only non singular matrices possess inverses.

Theorem :- The necessary and sufficient condition for a square matrix to possess inverse is that $|A| \neq 0$.

Note :- If $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

Find the inverse of $A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$

Sol: Given that $A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$

We have $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

$$|A| = \begin{vmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{vmatrix}$$

$$= 7(-6+4) - 2(0-3) + 1(0+9)$$

$$= -14 + 6 + 9$$

$$|A| = 1$$

Cofactor of an element $a_{11} = 7$ is $A_{11} = \begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}(-1)^{1+1} = -2$

Cofactor of an element $a_{12} = 2$ is $A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -1 \\ -3 & -2 \end{vmatrix} = 3$

Cofactor of an element $a_{13} = 1$ is $A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 3 \\ -3 & 4 \end{vmatrix} = 9$

Cofactor of an element $a_{21} = 0$ is $A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = 8$

Cofactor of an element $a_{22} = 3$ is $A_{22} = (-1)^{2+2}\begin{vmatrix} 7 & 1 \\ -3 & -2 \end{vmatrix} = -11$

Cofactor of an element $a_{23} = -1$ is $A_{23} = (-1)^{2+3}\begin{vmatrix} 7 & 2 \\ -3 & 4 \end{vmatrix} = -34$

Cofactor of an element $a_{31} = -3$ is $A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -5$

Cofactor of an element $a_{32} = 4$ is $A_{32} = (-1)^{3+2}\begin{vmatrix} 7 & 1 \\ 0 & -1 \end{vmatrix} = 7$

cofactor of an element $a_{33} = -2$ is $A_{33} = (-1)^{3+3}\begin{vmatrix} 7 & 2 \\ 0 & 3 \end{vmatrix} = 21$

$$\text{Cofactor matrix of } A = \begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}$$

$$\text{adj } A = [\text{Cofactor matrix of } A]^T = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

We have $A^{-1} = \dfrac{1}{|A|} \text{adj } A$

$$\therefore \quad A^{-1} = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

Matrix inversion Method :-

The system of linear equations are

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}\right\} \quad \text{--- } (1)$$

The matrix form of given system of equations is $AX = B$.

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The solution of the given system is $X = A^{-1}B$.

Solve $7x + 2y + z = 21$, $3y - z = 5$, $-3x + 4y - 2z = -1$, by Matrix inversion method.

Sol: Given that $7x + 2y + z = 21$

$3y - z = 5$

$-3x + 4y - 2z = -1$

The matrix form of given system of equations is $AX = B$.

where $A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix}$

The solution of system of equations by matrix inversion method is $X = A^{-1}B$

where $A^{-1} = \frac{1}{|A|} \text{adj } A$.

$$|A| = \begin{vmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{vmatrix} = 7(-6+4) - 2(0-3) + 1(0+9) = 1$$

$$\text{Cofactor matrix of } A = \begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}$$

$$\text{adj } A = \left[\text{Cofactor matrix of } A\right]^T = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

We have $A^{-1} = \dfrac{1}{|A|} \text{adj } A$

$$\therefore \quad A^{-1} = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

$$X = A^{-1} B$$

$$X = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix} \begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} -42 + 40 + 5 \\ 63 - 55 - 7 \\ 189 - 170 - 21 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Which is the solution of the given system of eqns.

CRAMER'S RULE (DETERMINANT METHOD) :-

The given system of linear equations are

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} \text{---} (1)$$

The matrix form of the system (1) is $AX = B$.

Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The solution of the system (1) is given by

$$x = \frac{\Delta_1}{\Delta} \quad y = \frac{\Delta_2}{\Delta} \quad z = \frac{\Delta_3}{\Delta} \quad (\Delta \neq 0)$$

Where $\Delta = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

We notice that $\Delta_1, \Delta_2, \Delta_3$ are the determinants obtained from Δ on replacing the 1st, 2nd and 3rd columns by d's i.e (d_1, d_2, d_3) respectively.

Solve $-x+3y-2z=5$, $4x-y-3z=-8$, $2x+2y-5z=7$ by Cramer's rule.

Sol:- Given that $-x+3y-2z=5$, $4x-y-3z=-8$ $2x+2y-5z=7$.

The matrix form of given System of eqn's is $AX = B$

Where $A = \begin{bmatrix} -1 & 3 & -2 \\ 4 & -1 & -3 \\ 2 & 2 & -5 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ -8 \\ 7 \end{bmatrix}$

The solution of linear system of equations by cramer's rule is given by $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$ $z = \frac{\Delta_3}{\Delta}$.

$$\Delta = |A| = \begin{vmatrix} -1 & 3 & -2 \\ 4 & -1 & -3 \\ 2 & 2 & -5 \end{vmatrix} = -1(5+6) - 3(-20+6) - 2(8+2)$$

$$\Delta = |A| = -11 + 42 - 20 = 11$$

$$\Delta_1 = \begin{vmatrix} 5 & 3 & -2 \\ -8 & -1 & -3 \\ 7 & 2 & -5 \end{vmatrix} = 5(5+6) - 3(40+21) - 2(-16+7)$$

$$\Delta_1 = 55 - 183 + 18 = -110$$

$$\Delta_2 = \begin{vmatrix} -1 & 5 & -2 \\ 4 & -8 & -3 \\ 2 & 7 & -5 \end{vmatrix} = -1(40+21) - 5(-20+16) - 2(28+16)$$

$$\Delta_2 = -61 + 70 - 88 = -79$$

$$\Delta_3 = \begin{vmatrix} -1 & 3 & 5 \\ 4 & -1 & -8 \\ 2 & 2 & 7 \end{vmatrix} = -1(-7+16) - 3(28+16) + 5(8+2)$$

$$\Delta_3 = -9 - 132 + 50$$

$$\Delta_3 = 91$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-110}{11} = -10 \qquad y = \frac{\Delta_2}{\Delta} = \frac{-79}{11} \qquad z = \frac{\Delta_3}{\Delta} = \frac{-91}{11}$$

$\therefore$ The solution of the given system of equations is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ -79/11 \\ -91/11 \end{bmatrix}$$

Sub Matrix :— A matrix obtained by deleting a row or a column or both of a given matrix is called its sub matrix of the given matrix.

Eg:- Let $A = \begin{bmatrix} 1 & 3 & -4 & 7 & 8 \\ 9 & 8 & 2 & 8 & 7 \\ 5 & 6 & 9 & 5 & 3 \end{bmatrix}_{3\times5}$

Then $\begin{bmatrix} 1 & 3 & 7 & 8 \\ 9 & 8 & 8 & 7 \\ 5 & 6 & 5 & 3 \end{bmatrix}$ is a sub matrix of A obtained by deleting third column from A.

Similarly $\begin{bmatrix} 1 & 3 & 8 \\ 9 & 8 & 7 \end{bmatrix}$ is a sub matrix of A obtained by deleting third row and 3rd, 4th column from A.

Minor of a matrix :—

Let A be an $m\times n$ matrix. The determinant of a square sub matrix of A is called a minor of the matrix. If the order of the square sub matrix is t then its determinant is called a minor of order t.

Eg:- $A = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 5 & 8 \\ 6 & 0 & 1 \end{bmatrix}_{4\times3}$ be a matrix.

We have $B = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$ be a sub matrix of order 2.

$|B| = 9 - 21 = -12$ is a minor of order 2.

Rank of a Matrix :-

Let A be an $m\times n$ matrix. If A is a null matrix. We define it Rank to be zero. If A is not null matrix. We say that r is the rank of A :

if (i) Every $(r+1)^{th}$ order minor of A is zero.

(ii) There exists atleast one r^{th} order minor of A which is not zero

Rank of A is denoted by $\rho(A)$.

Note :- (1) It can be noted that the rank of a non zero matrix is the order of the highest order non zero minor of A.

(2) Rank of a matrix is unique.

(3) Every matrix will have a rank.

(4) If A is a matrix of order $m\times n$ then

$$\text{Rank of } A = \rho(A) \le \min\{m,n\}$$

Eg:- $A = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 12 \end{bmatrix}_{2\times 3}$

Given matrix is of order 2×3.

$$\rho(A) \le \min\{2,3\}$$

$$\text{i.e } \rho(A) \le 2.$$

$\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$ be sub matrix of order 2 of the given matrix.

$$\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 9-21 = -12 \neq 0$$

$$\therefore \rho(A) = 2.$$

(5) If $\rho(A) = r$ then every minor of A of order $r+1$ or more is zero.

Eg:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 1(2-0) - 2(4-0) + 3(2-0) = 0$$

$|A| = 0$ i.e A is singular

$\Rightarrow \rho(A) < 3$

Consider the minor of order 2, $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = -3 \neq 0.$

$\therefore \rho(A) = 2$.

(6) Rank of the identity matrix I_n is n

Eg:- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\rho(I) = 2$.

(7) If A is non singular matrix of order n then $\rho(A) = n$.

Eg:- $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2 - 12 = -10 \neq 0$$

$\therefore |A| \neq 0$ i.e A is non singular

$\therefore \rho(A) = 2$.

(8) If A is a matrix, A^T is transpose of matrix A Then $\rho(A) = \rho(A^T)$

Eg:- $A = \begin{bmatrix} 1 & 2 & -5 \\ -3 & 4 & 6 \end{bmatrix}$

A is rectangular matrix of order 2×3.

$\rho(A) \leq \min\{2, 3\}$

$\rho(A) \leq 2$

Consider the minor of order 2, $\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} = 4+6 = 10 \neq 0$

$\therefore \rho(A) = 2$.

$$A^T = \begin{bmatrix} 1 & -3 \\ 2 & 4 \\ -5 & 6 \end{bmatrix}$$

A^T is rectangular matrix of order 3×2.

$\rho(A^T) \leq \min\{3,2\}$

$\rho(A^T) \leq 2$

Consider the minor of order 2, $\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} = 4+6 = 10 \neq 0$.

$\therefore \rho(A) = 2$.

$\therefore \rho(A) = \rho(A^T)$

(9) If A is singular matrix of order n then $\rho(A) < n$.

Eg :- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4-4 = 0$.

A is singular matrix

$\rho(A) < 2$.

A is not null matrix

$\therefore \rho(A) = 1$

(10) The Rank of non zero row matrix is 1.

Eg :- $A = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \end{bmatrix}_{1\times 5}$

$\rho(A) = 1$.

(11) The Rank of non zero column matrix is 1.

Eg :- $A = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}_{4\times 1}$ $\rho(A) = 1$.

(12) The rank of matrix is $\geq r$ if there is atleast one minor of r^{th} order which is not equal to zero.

→ Find the value of k such that the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2.

Sol: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Given that $\rho(A) = 2$.

So every minor of order greater than 2 is zero

i.e $|A| = 0$ i.e $\begin{vmatrix} 1 & 2 & 7 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{vmatrix} = 0$

$$1(10k-42) - 2(20-21) + 7(12-3k) = 0$$

$$\therefore k = 4$$

→ Find the rank of a matrix $A = \begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$

Sol: Given that $A = \begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{bmatrix}_{3\times 4}$

A is rectangular matrix of order 3×4

$\rho(A) \leq \min\{3,4\}$

$\rho(A) \leq 3$.

Consider the minor of order 3, $\begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} =$

$$= 0 - 1(0-3) - 3(1) = 0.$$

Consider the minor of order 3, $\begin{vmatrix} 1 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} =$

$$1(2-0) + 3(0-1) + 1(0-1) = -2 \neq 0.$$

One minor of order 3 is not zero

$\therefore \rho(A) = 3$.

→ Find the rank of matrix $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

sol: G/T $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

A is a square matrix of order 3

$\rho(A) \leq 3$

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{vmatrix} = 3(4-4)+1(-12+12)+2(-6+6) = 0$$

Consider the minor of order 2, $\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} = -4-4 = -8 \neq 0$.

One minor of order 2 is not equal to zero.

$\therefore \rho(A) = 2$.

→ Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

sol: G/T $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

A is a square matrix of order 3

$\rho(A) \leq 3$.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{vmatrix} = 1(48-40)-2(36-28)+3(30-28)$$

$$= 8 - 16 + 6 = -2 \neq 0$$

$|A| \neq 0$

$\therefore \rho(A) = 3$.

→ Find the values of x suchthat matrix $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singu-lar.

Sol:- Given that $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & 4 & -1-x \end{bmatrix}$

A is singular $\Rightarrow |A| = 0$

$$\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & 4 & -1-x \end{vmatrix} = 0$$

$R_2 \longrightarrow R_2 + R_3$

$$\begin{vmatrix} 3-x & 2 & 2 \\ 0 & -x & -x \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

$$-x \begin{vmatrix} 3-x & 2 & 2 \\ 0 & 1 & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

$C_3 \longrightarrow C_3 - C_2$

$$-x \begin{vmatrix} 3-x & 2 & 0 \\ 0 & 1 & 0 \\ -2 & -4 & -3-x \end{vmatrix} = 0$$

Expand it by using 3^{rd} column

$$x(3-x)^2 = 0$$

$$x = 0, 3.$$

Elementary transformations or operations on a matrix :-

(a) There are three types of elementary row operations.

(i) Interchange of two rows :- If i^{th} row and j^{th} row are interchanged, it is denoted by $R_i \leftrightarrow R_j$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$\begin{bmatrix} 1 & 0 & 7 \\ 4 & 6 & 3 \\ 2 & 5 & -3 \end{bmatrix}$

(ii) Multiplication of each element of a row with non zero scalar :-

If i^{th} row is multiplied with k then it is denoted by $R_i \to R_i(k)$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$R_2 \to 2R_2$

$\begin{bmatrix} 1 & 0 & 7 \\ 4 & 10 & -6 \\ 4 & 6 & 3 \end{bmatrix}$

(iii) Multiplying every element of a row which is a non zero scalar and adding to the corresponding elements of another row :-

If the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by

$R_j \to R_j + kR_i$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 5 & -17 \\ 4 & 6 & 3 \end{bmatrix}$

(b) There are three types of elementary column operations.

(i) Interchange of two columns :- If i^{th} column and j^{th} column are interchanged, it is denoted by $C_i \leftrightarrow C_j$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$C_1 \leftrightarrow C_2$

$\sim \begin{bmatrix} 0 & 1 & 7 \\ 5 & 2 & -3 \\ 6 & 4 & 3 \end{bmatrix}$

(ii) Multiplication of each element of a column with a non zero scalar :-

If i^{th} row is multiplied with k then it is denoted by $C_i \to C_i(k)$.

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$C_2 \to C_2(2)$

$\sim \begin{bmatrix} 1 & 0 & 7 \\ 2 & 10 & -3 \\ 4 & 12 & 3 \end{bmatrix}$

(iii) Multiplying every element of a column which is a non zero scalar and adding to the corresponding elements of another column :-

If the elements of i^{th} column are multiplied with k and added to the corresponding elements of j^{th} column then it is denoted by.

$C_j \to C_j + kC_i$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$C_3 \to C_3 - 7C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & -17 \\ 4 & 6 & -25 \end{bmatrix}$

Equivalence of Matrices :-

If a matrix B is obtained from a matrix A after a finite chain of elementary transformations then B is said to be equivalent to A.

Symbolically it is denoted as $B \sim A$

Eg:- $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 5 & 9 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 5 & 9 \end{bmatrix} = B$$

Matrix B obtained from a matrix A after elementary row transformation. So the matrix B is said to be equivalent to A.

Zero row and Non zero row :-

If all the elements in a row of a matrix are zero's then it is called zero row and if there is atleast one zero element in a row then it is called a non zero row.

Eg:- $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 9 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \rightarrow \text{Non zero row} \\ \} \rightarrow \text{zero row.} \end{matrix}$$

Echelon form of a matrix :-

A Matrix is said to be Echelon form if the following three properties are satisfied.

(i) zero rows if any must be below the non zero rows.

(ii) The first non zero element of a non zero row is equal to one

(iii) The no. of zeros before the non zero element of a row is less than such zeros in the next row.

Note :- The condition (ii) is not compulsory.

Result :- The no. of non zero rows in a echelon form of A is the rank of A.

Eg :- $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Is in echelon form since it satisfies all the three conditions of the echelon form

$\therefore \rho(A) = 3 =$ No. of non zero rows.

Working procedure to reduce a matrix into echelon form :-

Case (i) :-

Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Step 1 :- If $a_{11} \neq 0$, by using a_{11} position, make a_{21} and a_{31} positions as zero. Here we apply row operations only.

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \end{bmatrix}$$

Step 2 :- If $a'_{22} \neq 0$, by using a'_{22} position, make a'_{32} position as zero. Here we apply row operations only.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \end{bmatrix}$$

Which is in echelon form.

$\rho(A) = 3$ if $a''_{33} \neq 0$ or $a''_{34} \neq 0$.

(or) $\rho(A) = 2$ if $a''_{33} = 0$ and $a''_{34} = 0$.

Case (ii) :- Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

Step 1 :- If $a_{11} \neq 0$, by using a_{11} position make a_{21}, a_{31} and a_{41} positions as zero. Here we apply row operations only.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix}$$

Step 2 :- If $a'_{22} \neq 0$, by using a'_{22} position make a'_{32} and a'_{42} positions as zero. Here we apply row operations only.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & a''_{43} & a''_{44} \end{bmatrix}$$

Step 3 :- If $a''_{33} \neq 0$ by using a''_{33} position make a''_{43} position as zero. Here we apply row operation only.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix}$$

Which is in echelon form

$\rho(A) = 4$ if $a'''_{44} \neq 0$

(or) $\rho(A) = 3$ if $a'''_{44} = 0$.

→ Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ by reduce it to echelon form.

Sol:- Given that $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Now we reduce the matrix A into echelon form by applying eleme-ntary row operations only.

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 2R_1, \quad R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$R_4 \to 3R_4 - 2R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is in echelon form.

$\rho(A)$ = No. of non zero rows of the last equivalent to A $= 4$

$\therefore \rho(A) = 4$.

→ Show that the equations $x-3y-8z=-10$, $3x+y-4z=0$, $2x+5y+6z=13$ are consistent and solve the same.

Sol: Given that $x-3y-8z=-10$, $3x+y-4z=0$, $2x+5y+6z=13$

These are $m=3$ eqn's in $n=3$ unknowns x, y, z.

The matrix equation of the given system of eqn's is $AX=B$

Where $A=\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix}$ $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B=\begin{bmatrix} -10 \\ 0 \\ 13 \end{bmatrix}$

The augmented matrix $[A|B]=\left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{array}\right]$

Now reduce the augmented matrix $[A|B]$ to echelon form by using E-row operations only and determine the $\rho(A)$ and $\rho([A|B])$ respectively.

$R_2 \to R_2-3R_1$, $R_3 \to R_3-2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{array}\right]$$

$R_3 \to 10R_3-11R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which is in echelon form.

Hence $\rho(A)=2=$ The no. of non zero rows of equivalent A.

$\rho([A|B])=2=$ The no. of non zero rows of equivalent to $[A|B]$

$\rho(A)=\rho([A|B])=2<3$ (No. of unknowns)

So that the system is consistent and possesses an infinite no. of sol's.

To determine these solutions we have to assign arbitrary values

→ Discuss for what values of λ, μ the simultaneous equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (i) No solution. (ii) a unique solution (iii) an infinite no. of solutions.

Sol:- Given that $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$.

There are $m=3$ equations in $n=3$ unknowns x, y and z.

The matrix form of the given system of equations is $AX=B$.

Where $A=$ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

The augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

Now reduce the augmented matrix $[A|B]$ to echelon form by using E-row operations only and determine ranks of A and $[A|B]$ respectively.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

Which is in echelon form.

Case(i):- No solution

Suppose $\lambda=3$ and $\mu\neq 0$. then $\rho(A)=2$ and $\rho([A|B])=3$

$\rho(A) \neq \rho([A|B])$

$\therefore$ The system is inconsistent

$\therefore$ It has no solution.

Now the equivalent matrix eqn. of $AX = B$ is .

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -8 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\beta \\ \tan\gamma \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

The corresponding system of eqns is

$$2\sin\alpha - \cos\beta + 3\tan\gamma = 3$$

$$4\cos\beta - 8\tan\gamma = -4$$

$$-8\tan\gamma = 0. \Rightarrow \gamma = 0$$

$$\cos\beta = \frac{-4 + \tan\gamma}{4}$$

$$\cos\beta = -1 \Rightarrow \beta = \pi$$

$$\sin\alpha = \frac{3 + \cos\beta - 3\tan\gamma}{2}$$

$$\sin\alpha = 1 \Rightarrow \alpha = \frac{\pi}{2}.$$

Hence $\alpha = \frac{\pi}{2}$, $\beta = \pi$ and $\gamma = 0$ is the sol. of the system.

SYSTEM OF NON HOMOGENEOUS LINEAR EQUATIONS

5

1 Test for consistency and hence solve $x+y+z=6$, $x-y+2z=5$, $3x+y+z=8$, $2x-2y+3z=7$ Ans:- $x=1$ $y=2$ $z=3$.

2 Test for consistency $3x+3y+2z=1$, $x+2y-4=0$, $10y+3z=-2$, $2x-3y-z=5$ Ans:- $x=2$ $y=1$ $z=-4$.

3 If consistent, solve $x+y+z+t=4$, $x-z+2t=2$, $y+z-3t=-1$, $x+2y-z+t=3$.
Ans:- $x=y=z=t=1$.

4 Solve completely the equations $3x-2y-w=2$, $2y+2z+w=1$, $y+2z+w=1$, $x-2y-3z+2w=3$ Ans:- $x=w=1$, $y=z=0$.

5 Show that the equations $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$, $x-y+z=-1$ are consistent and solve them $x=-1$, $y=4$, $z=4$.

6 Solve the system for x, y and z, $-\frac{1}{x}+\frac{3}{y}+\frac{4}{z}=30$, $\frac{3}{x}+\frac{2}{y}+\frac{1}{z}=9$, and $\frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10$. Ans: $x=\frac{1}{2}$, $y=\frac{1}{4}$, $z=\frac{1}{5}$.

7 Solve the following system of non linear equations for the unknown angles α, β and γ where $0\leq\alpha\leq 2\pi$, $0\leq\beta\leq 2\pi$ and $0\leq\gamma<\pi$.
$2\sin\alpha-\cos\beta+3\tan\gamma=3$, $4\sin\alpha+2\cos\beta-2\tan\gamma=2$, $6\sin\alpha-3\cos\beta+\tan\gamma=9$.
Ans:- $\alpha=\frac{\pi}{2}$, $\beta=\pi$, $\gamma=0$.

8 Determine the values of λ for which the system $3x-y+\lambda z=0$, $2x+y+z=2$, $x-2y-\lambda z=-1$ will fail to have a unique solution. For what value of λ are the equations consistent. Ans:- $\lambda=-\frac{7}{2}$, No Solution.

9 For what values of a and b the equations $x+2y+3z=8$, $2x+y+3z=13$, $3x+4y-az=b$ have (i) No solution (ii) A unique solution (iii) An infinite no. of solutions.

10. Solve the system if consistent $x+y+z=-3$, $3x+y-2z=-2$, $2x+4y+7z=7$ are inconsistent.

SYSTEM OF NON HOMOGENEOUS LINEAR EQUATIONS.

1 Are the following equations consistent, if so solve them.

$x_1 - x_2 + x_3 - x_4 + x_5 = 1$, $2x_1 - x_2 + 3x_3 + 4x_5 = 2$, $3x_1 - 2x_2 + 2x_3 + x_4 + x_5 = 1$.

$x_1 + x_3 + 2x_4 + x_5 = 0$. Ans:- $x_4 = K_1$ $x_5 = K_2$, $x_3 = 1 + K_1 - 2K_2$,

$x_2 = -1 - 3K_1$, $x_1 = -1 + 3K_1 + K_2$.

2 Solve the system completely $x+y+z=1$, $x+2y+4z=\alpha$, $x+4y+10z=\alpha^2$.

Ans:- $\alpha = 1$, $x = 1+2K_1$, $y = -3K_1$, $z = K_1$; $\alpha = 2$, $x = 2K_2$, $y = 1-3K_2$, $z = K_2$.

3 show that the equations $-2x+y+z=a$, $x-2y+z=b$, $x+y-2z=c$ have no solution unless $a+b+c=0$, in which case, they have infinitely many solution. Find these solutions when $a=1$, $b=1$, $c=-2$.

Ans:- $x = K-1$, $y = K-1$, $z = K$.

4 Find for what values of λ, the set of equations $2x-3y+6z-5t=3$, $y-4z+t=1$, $4x-5y+8z-9t=\lambda$ has (i) No solution (ii) Infinite number of solutions and find the solution of the equations when they are consistent.

Ans:- (i) $\lambda \neq 7$ (ii) $\lambda = 7$, $x = 3K_1 + K_2 + 3$, $y = 4K_1 - K_2 + 1$, $z = K_1$, $t = K_2$

5 show that if $\lambda \neq 0$, the system of equations $2x+y=a$, $x+\lambda y - z = b$, $y+2z=c$ has a unique solution for every value of a, b, c. If $\lambda = 0$, determine the relation satisfied by a, b, c such that the system is consistent. Find the solution by taking $\lambda=0$, $a=1$, $b=1$, $c=-1$. Ans:- $x = 1+K_1$, $y = -1-2K_1$, $z = K_1$

6 Find the value of λ for which the system of equations $3x-y+4z=3$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$ will have infinite number of solutions and solve them with the same λ value. Ans:- $x = \frac{4-5K}{7}$, $y = \frac{13K-9}{7}$, $z = K$.

7 show that the equations $4x-y+6z=16$, $x-4y-3z=-16$, $2x+7y+12z=48$, $5x-5y+3z=0$ are consistent and solve the same Ans:- $z=K$, $y = \frac{16}{3} - \frac{16}{5}K$, $x = \frac{16}{3} - \frac{9}{5}K$.

8 Solve $u+2v+2w=1$, $2u+v+w=2$, $3u+2v+2w=3$, $v+w=0$ Ans:- $u=1$, $v=-c$, $w=c$.

Method of Factorization [L-U De Composition Method] :

(Triangularisation):-

This method is based on the fact that a square matrix A can be factorized into the form LU where L is the unit lower triangular matrix and U is the upper triangular matrix. Here all principal minors of A must be non singular. This factorisation if it exists, is unique.

Consider a system of linear equations
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Which can be written in the matrix form $AX = B$ — ①.

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Let $A = LU$ — ②.

Where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ — ③ is the unit lower triangular matrix.

$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ is an upper triangular matrix.

Then from ① and ② $LUX = B$ — ③

Put $UX = Y$ — ④ Where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then ③ can be written as $LY = B$ — ⑤.

$$⑤ \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$$y_1 = b_1$$
$$l_{21}y_1 + y_2 = b_2$$
$$l_{31}y_1 + l_{32}y_2 + y_3 = b_3$$

This can be solved for y_1, y_2, y_3 by forward substitution.

Then (4) $\Rightarrow UX = Y$.

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$u_{22}x_2 + u_{23}x_3 = y_2$$
$$u_{33}x_3 = y_3.$$

Which can be solved for $x_1, x_2,$ and x_3 by backward substitution.

<u>Computation of Lower and Upper triangular matrices L and U</u>:—

From equation (2) we have

$$LU = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now equating the corresponding elements on both sides, we get

$$u_{11} = a_{11} \qquad u_{12} = a_{12} \qquad u_{13} = a_{13}$$

$$l_{21}u_{11} = a_{21} \Rightarrow l_{21} = \frac{a_{21}}{u_{11}} = \frac{a_{21}}{a_{11}}.$$

$$l_{31} u_{11} = a_{31} \implies l_{31} = \frac{a_{31}}{u_{11}} = \frac{a_{31}}{a_{11}}$$

$$l_{21} u_{12} + u_{22} = a_{22} \implies u_{22} = a_{22} - l_{21} u_{12}$$

$$u_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

$$l_{21} u_{13} + u_{23} = a_{23} \implies u_{23} = a_{23} - l_{21} u_{13}$$

$$u_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13}$$

$$l_{31} u_{12} + l_{32} u_{22} = a_{32} \implies l_{32} = \frac{a_{32} - l_{31} u_{12}}{u_{22}}$$

$$l_{32} = \frac{a_{32} - \left(\frac{a_{31}}{a_{11}}\right) a_{12}}{a_{22} - \left(\frac{a_{21}}{a_{11}}\right) a_{12}}$$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = a_{33}$ from which u_{33} can be calculated.

We have a systamatic procedure to evaluate the elements of L and U.

Step 1 :- We determine the first row of U and the first column of L.

Step 2 :- We determine the second row of U and the second column of L

Step 3 :- Finally we compute the third row of U. This procedure can be obviously generalized. This method is also called as L-U decomposition method.

(1) Solve the system of equations $2x+3y+z=9$, $x+2y+3z=6$, $3x+y+2z=8$ by the factorization method.

sol:- Given that
$$\left.\begin{aligned} 2x+3y+z&=9\\ x+2y+3z&=6\\ 3x+y+2z&=8 \end{aligned}\right\} \longrightarrow (1)$$

The matrix form of the given system of eqns is $AX=B \longrightarrow (2)$

where $A=\begin{bmatrix} 2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}$ $X=\begin{bmatrix} x\\ y\\ z\end{bmatrix}$ $B=\begin{bmatrix} 9\\ 6\\ 8\end{bmatrix}$

Let $A=LU \longrightarrow (2)$

where $L=\begin{bmatrix} 1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$ is the unit lower triangular matrix

$U=\begin{bmatrix} u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$ is the upper triangular matrix.

From (1) and (2), $LUX=B \longrightarrow (3)$.

Taking $UX=Y \longrightarrow (4)$ where $Y=\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix}$

From (3) and (4), $LY=B \longrightarrow (5)$

<u>To find the matrices L and U</u> :—

From equation (2) we have $LU=A$

$$\begin{bmatrix} 1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix} = \begin{bmatrix} 2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Equating the corresponding elements bothsides, we get

$u_{11} = 2 \qquad u_{12} = 3 \qquad u_{13} = 1$

$l_{21}u_{11} = 1 \implies l_{21} = \frac{1}{2}$

$l_{31}u_{11} = 3 \implies l_{31} = \frac{3}{2}$.

$l_{21}u_{12} + u_{22} = 2 \implies \frac{3}{2} + u_{22} = 2 \quad$ i.e $u_{22} = 2 - \frac{3}{2} = \frac{1}{2}$.

$l_{21}u_{13} + u_{23} = 3 \implies \frac{1}{2} + u_{23} = 3 \quad$ i.e $u_{23} = 3 - \frac{1}{2} = \frac{5}{2}$.

$l_{31}u_{12} + l_{32}u_{22} = 1 \implies \frac{9}{2} + l_{32}\frac{1}{2} = 1 \quad$ i.e $l_{32} = -7$.

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$

$$\implies \frac{3}{2} - \frac{35}{2} + u_{33} = 2$$

$$u_{33} = 2 - \frac{3}{2} + \frac{35}{2}$$

$$u_{33} = 18$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

From equation (5), First we have to find the values of y_1, y_2 and y_3.

i.e $LY = B \implies \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

$$y_1 = 9$$

$$\frac{1}{2}y_1 + y_2 = 6$$

$$\frac{3}{2}y_1 - 7y_2 + y_3 = 8$$

solving the above equations by forward substitution.

$$y_2 = 6 - \frac{1}{2}y_1 = 6 - \frac{1}{2}9 = \frac{3}{2}$$

$$y_3 = 8 - \frac{3}{2}y_1 + 7y_2 = 8 - \frac{27}{2} + \frac{21}{2}$$

$$y_3 = 5$$

$$\therefore \; y_1 = 6 \quad y_2 = \frac{3}{2} \quad y_3 = 5.$$

From the equation (4), we have to find the values of x, y and z.

$$UX = Y \implies \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$$2x + 3y + z = 9$$

$$\frac{1}{2}y + \frac{5}{2}z = \frac{3}{2}$$

$$18z = 5$$

solving the above eqn's by backward substitution.

$$z = \frac{5}{18}$$

$$\frac{5}{2}z = \frac{3}{2} - \frac{1}{2}y \quad \text{(OR)} \quad \frac{y}{2} = \frac{3}{2} - \frac{5z}{2}$$

$$\frac{y}{2} = \frac{3}{2} - \frac{5}{2}\cdot\frac{5}{18}$$

$$y = 3 - \frac{25}{18} = \frac{29}{18}$$

$$z = 9 - 2x - 3y \quad \text{(OR)} \quad 2x = 9 - 3y - z$$

$$2x = 9 - 3\cdot\frac{29}{18} - \frac{5}{18} = \frac{70}{18}$$

$$x = \frac{35}{18}.$$

$\therefore$ The solution of the given system is $x = \frac{35}{18}$, $y = \frac{29}{18}$, $z = \frac{5}{18}$.

→ Solve the system $x+2y+3z=10$, $3x+y+2z=13$, $2x+3y+z=13$ by LU Decomposition Method.

Sol:- Given that $x+2y+3z=10$, $3x+y+2z=13$, $2x+3y+z=13$ —— (1)

The matrix form of the given system of eqn's is $AX=B$ —— (1)

Where $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 2 & 3 & 1\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ $B=\begin{bmatrix}10\\ 13\\ 13\end{bmatrix}$

Step (i) :- Let $A=LU$ —— (2)

where $L=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$ is the unit lower triangular matrix

$U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$ is the upper triangular matrix.

From (1) and (2), $LUX=B$ —— (3)

Taking $UX=Y$ —— (4) where $Y=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$

From (3) and (4), $LY=B$ —— (5)

Step (ii) :- To find the matrices L and U :-

From equation (2), we have $LU=A$

$$\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 3 & 1\end{bmatrix}$$

$$\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 2\\ 2 & 3 & 1\end{bmatrix}$$

Equating the corresponding elements bothsides, we get

$$u_{11} = 1 \qquad u_{12} = 2 \qquad u_{13} = 3.$$

$$l_{21} u_{11} = 3 \implies l_{21} = 3.$$

$$l_{31} u_{11} = 2 \implies l_{31} = 2$$

$$l_{21} u_{12} + u_{22} = 1 \implies u_{22} = 1 - l_{21} u_{12}$$

$$u_{22} = 1 - 3(2) = -5$$

$$l_{21} u_{13} + u_{23} = 2 \implies u_{23} = 2 - l_{21} u_{13}$$

$$u_{23} = 2 - 3(3) = -7.$$

$$l_{31} u_{12} + l_{32} u_{22} = 3 \implies l_{32} = \frac{3 - l_{31} u_{12}}{u_{22}} = \frac{3-4}{-5} = \frac{1}{5}$$

$$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 1 \implies u_{33} = 1 - l_{31} u_{13} - l_{32} u_{23}$$

$$u_{33} = 1 - 6 + \frac{7}{5} = \frac{-18}{5}$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{5} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & \frac{-18}{5} \end{bmatrix}$$

step(iii) :- From equation (5) first we have to find the values of y_1, y_2 and y_3.

$$\text{i.e} \quad LY = B \implies \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 13 \end{bmatrix}$$

$$y_1 = 10$$

$$3y_1 + y_2 = 13$$

$$y_2 = 13 - 3y_1 = 13 - 30$$

$$y_2 = -17.$$

$$2y_1 + \frac{1}{5} y_2 + y_3 = 13.$$

$$y_3 = 13 - 2y_1 - \frac{1}{5} y_2$$

$$y_3 = \frac{-18}{5}$$

$$\therefore\ Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -17 \\ -\frac{18}{5} \end{bmatrix}$$

<u>step (iv)</u> :- From the equation ④ we have to find the values of x, y and z.

$$UX = Y \implies \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -\frac{18}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -17 \\ -\frac{18}{5} \end{bmatrix}$$

$$x + 2y + 3z = 10$$

$$-5y - 7z = -17. \implies 5y + 7z = 17$$

$$-\frac{18}{5} z = -\frac{18}{5} \implies z = 1.$$

$$y = \frac{17 - 7z}{5} = \frac{17-7}{5} = 2$$

$$x = 10 - 2y - 3z$$

$$x = 10 - 4 - 3$$

$$x = 3.$$

$$\therefore\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$ is the solution of the given system.

→ Solve $-3x+12y-6z=-33$; $x-2y+2z=7$, $y+z=-1$ using LU-Decomposition Method.

Sol:- Given that $-3x+12y-6z=-33$, $x-2y+2z=7$, $y+z=-1$

The matrix form of the given system is $AX=B$ —— ①.

where $A=\begin{bmatrix}-3 & 12 & -6\\ 1 & -2 & 2\\ 0 & 1 & 1\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ $B=\begin{bmatrix}-33\\ 7\\ -1\end{bmatrix}$

Step (i) :- Let $A=LU$ —— ②

Where $L=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$ is the unit lower triangular matrix

$U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$ is the upper triangular matrix.

From ① and ②, $LUX=B$ —— ③

Taking $UX=Y$ —— ④ where $Y=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$

From ③ and ④, $LY=B$ —— ⑤

Step (ii) :- To find the matrices L and U :—

From equation ②, we have $LU=A$

$$\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix}-3 & 12 & -6\\ 1 & -2 & 2\\ 0 & 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}-3 & 12 & -6\\ 1 & -2 & 2\\ 0 & 1 & 1\end{bmatrix}$$

Equating the corresponding elements bothsides, we get

$$u_{11} = -3 \qquad u_{12} = 12 \qquad u_{13} = -6 .$$

$$l_{21} u_{11} = 1 \implies l_{21} = -\frac{1}{3}.$$

$$l_{31} u_{11} = 0 \implies l_{31} = 0 .$$

$$l_{21} u_{12} + u_{22} = -2 \implies u_{22} = -2 - l_{21} u_{12}$$

$$u_{22} = -2 + \frac{1}{3}(12) = 2.$$

$$l_{21} u_{13} + u_{23} = 2 \implies u_{23} = 2 - l_{21} u_{13} = 2 - \left(\frac{-1}{3}\right)(-6)$$

$$u_{23} = 0 .$$

$$l_{31} u_{12} + l_{32} u_{22} = 1 \implies l_{32} = \frac{1 - l_{31} u_{12}}{u_{22}} = \frac{1-0}{2} = \frac{1}{2}.$$

$$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 1 \implies u_{33} = 1 - l_{31} u_{13} - l_{32} u_{23}$$

$$u_{33} = 1 - 0 = 1.$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} -3 & 12 & -6 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step (iii) :- From equation ⑤ first we have to find the values of y_1, y_2 and y_3.

$$\text{i.e } LY = B \implies \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$y_1 = -33$$

$$-\frac{1}{3} y_1 + y_2 = 7$$

$$\frac{1}{2} y_2 + y_3 = -1.$$

$$y_2 = 7 + \frac{1}{3} y_1 = 7 + \frac{1}{3}(-33) = -4$$

$$y_3 = -1 - \frac{1}{2} y_2 = -1 - \frac{1}{2}(-4) = 1.$$

$$\therefore \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ -4 \\ 1 \end{bmatrix}.$$

Step (iv) :- From the equation ④, we have to find the values of x, y and z

$$UX = Y \implies \begin{bmatrix} -3 & 12 & -6 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -33 \\ -4 \\ 1 \end{bmatrix}$$

$$-3x + 12y - 6z = -33$$

$$2y = -4 \implies y = -2$$

$$z = 1.$$

$$3x = 33 + 12y - 6z$$

$$x = \frac{33 + 12y - 6z}{3}$$

$$x = \frac{33 - 24 - 6}{3} = 1.$$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$ is the solution of the given system.

CROUT'S METHOD :-

Consider the linear system
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \text{—— (1)}$$

Which can be written in the matrix form $AX = B$ —— (2).

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Let $A = LU$ —— (3)

Where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Here L is the Lower triangular ~~for~~ matrix.

U is the unit upper triangular matrix.

Then from (2) and (3), $LUX = B$ —— (4).

Put $UX = Y$ —— (5) Where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Then (4) can be written as $LY = B$ —— (6).

$$(6) \Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$l_{11}y_1 = b_1$$

$$l_{21}y_1 + l_{22}y_2 = b_2$$

$$l_{31}y_1 + l_{32}y_2 + l_{33}y_3 = b_3.$$

This can be solved for y_1, y_2, y_3 by forward substitution.

Then (5) $\Rightarrow$ $UX = Y$

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

$$x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$x_2 + u_{23}x_3 = y_2$$
$$x_3 = y_3.$$

Which can be solved for x_1, x_2, x_3 and by backward substitution.

Computation of Lower and Upper triangular Matrices :—

We have. $LU = A$.

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{23}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now equating the corresponding elements on both sides, we get

$l_{11} = a_{11} \qquad l_{11}u_{12} = a_{12} \qquad l_{11}u_{13} = a_{13}$

$l_{21} = a_{21} \qquad l_{21}u_{12} + l_{22} = a_{22} \qquad l_{21}u_{23} + l_{22}u_{23} = a_{23}$

$l_{31} = a_{31} \qquad l_{31}u_{12} + l_{32} = a_{32} \qquad l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}$

From this, we obtain $u_{12}, u_{13}, u_{23}, l_{22}, l_{32}, l_{33}$ and thus L and U are obtained.

Use crout's method to solve the system $x+y+z=1$ $3x+y-3z=5$ $x-2y-5z=10$.

Sol:- Given that $x+y+z=1$ $3x+y-3z=5$ $x-2y-5z=10$.

The matrix form of the given system is $AX=B$ —— (1)

Where $A=\begin{bmatrix}1&1&1\\3&1&-3\\1&-2&-5\end{bmatrix}$ $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ $B=\begin{bmatrix}1\\5\\10\end{bmatrix}$

Let $A=LU$ —— (2).

Where $L=\begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}$ is the lower triangular matrix.

$U=\begin{bmatrix}1&u_{12}&u_{13}\\0&1&u_{23}\\0&0&1\end{bmatrix}$ is the unit upper triangular matrix.

From (1) and (2), $LUX=B$ —— (3).

Taking $UX=Y$ —— (4) Where $Y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

From (3) and (4), $LY=B$ —— (5).

To find the matrices L and U :—

From equation (2), we have $LU=A$.

$$\begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}\begin{bmatrix}1&u_{12}&u_{13}\\0&1&u_{23}\\0&0&1\end{bmatrix}=\begin{bmatrix}1&1&1\\3&1&-3\\1&-2&-5\end{bmatrix}$$

$$\begin{bmatrix}l_{11}&l_{11}u_{12}&l_{11}u_{13}\\l_{21}&l_{21}u_{12}+l_{22}&l_{21}u_{13}+l_{22}u_{23}\\l_{31}&l_{31}u_{12}+l_{32}&l_{31}u_{13}+l_{32}u_{23}+l_{33}\end{bmatrix}=\begin{bmatrix}1&1&1\\3&1&-3\\1&-2&-5\end{bmatrix}$$

Equating the corresponding elements bothsides, we get

$l_{11}=1$

$l_{11}u_{12}=1 \implies u_{12}=1.$

$l_{11}u_{13}=1 \implies u_{13}=1.$

$l_{21} = 3.$

$l_{21} u_{12} + l_{22} = 1 \implies l_{22} = 1 - l_{21} u_{12}$

$l_{22} = 1 - 3.(1) = -2.$

$l_{21} u_{13} + l_{22} u_{23} = -3 \implies u_{23} = \dfrac{-3 - l_{21} u_{13}}{l_{22}}$

$u_{23} = \dfrac{-3 - 3(1)}{-2} = 3.$

$l_{31} = 1$

$l_{31} u_{12} + l_{32} = -2. \implies l_{32} = -2 - l_{31} u_{12}$

$= -2 - 1(1) = -3$

$l_{32} = -3.$

$l_{31} u_{13} + l_{32} u_{23} + l_{33} = -5$

$\implies l_{33} = -5 - l_{31} u_{13} - l_{32} u_{23}$

$l_{33} = -5 - 1(1) - (-3)(3) = -5 - 1 + 9 = 3.$

$l_{33} = 3.$

$$\therefore \quad L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -3 & 3 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

From equation (5), First we have to find the values of y_1, y_2, and y_3.

i.e $LY = B \implies \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$

$y_1 = 1$

$3y_1 - 2y_2 = 5$

$y_1 - 3y_2 + 3y_3 = 10.$

solving the above equations by forward substitution.

$y_2 = \dfrac{3y_1 - 5}{2} \implies y_2 = -1.$

$$y_3 = \frac{10 + 3y_2 - y_1}{3}$$

$$y_3 = \frac{10 - 3 - 1}{3} = 2 .$$

From the equation (4), we have to find the values of x, y and z.

$$UX = Y \implies \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$x + y + z = 1$$
$$y + 3z = -1$$
$$z = 2$$

Solving the above equation by backward substitution.

$$z = 2$$
$$y = -1 - 3z = -7.$$
$$x = 1 - y - z = 1 + 7 - 2 = 6.$$

$\therefore x = 6, \quad y = -7 \quad z = 2$.

Which is the required solution of the given system.

Solution to Tri-diagonal Systems :-

Definition :- If the coefficient matrix of a system of linear equations ie $AX = B$ has non zero elements along the main diagonal and the adjacent diagonals on either side of the main diagonal, then the system is called a "Tri diagonal system".

Working procedure :-

Consider the system of equations.

$$a_{11} x_1 + a_{12} x_2 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$

$$a_{32} x_2 + a_{33} x_3 + a_{34} x_4 = b_3$$

$$a_{43} x_3 + a_{44} x_4 = b_4 .$$

Step 1 :- The matrix equation of the given tri diagonal system is

$$AX = B \quad \text{--- (1)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ \text{[illegible]} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{bmatrix}$ is the coefficient matrix of the system.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is the matrix of unknowns $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ is the constant matrix.

Step 2 :- Let $A = LU$ --- (2).

Where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}$ is the unit lower triangular matrix.

$U = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$ is an upper triangular matrix.

From (1) and (2), $LUX = B$ --- (3).

Step 3 :- Put $UX = Y$ — ④ Where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

From ③ $LY = B$.

i.e $\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

The linear equations are

$$y_1 = b_1$$
$$l_{21} y_1 + y_2 = b_2$$
$$l_{32} y_2 + y_3 = b_3$$
$$l_{43} y_3 + y_4 = b_4$$

This can be solved for y_1, y_2 and y_3, y_4 by forward substitution.

Step 4 :- Using ④ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$, we get

$$UX = Y \Rightarrow \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

The linear equations are

$$u_{11} x_1 + u_{12} x_2 = y_1$$
$$u_{22} x_2 + u_{23} x_3 = y_2$$
$$u_{33} x_3 + u_{34} x_4 = y_3$$
$$u_{44} x_4 = y_4$$

Which can be solved for x_1, x_2, x_3 and x_4 by backward substitution.

Thus when L and U are known, we can calculate y_1, y_2, y_3, y_4 and x_1, x_2, x_3, x_4 by the above process.

Computation of L and U :-

We have $A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & u_{23} & 0 \\ 0 & l_{32}u_{22} & l_{32}u_{23}+u_{33} & u_{34} \\ 0 & 0 & l_{43}u_{33} & l_{43}u_{34}+u_{44} \end{bmatrix}$$

Equating the corresponding elements on bothsides.

$u_{11} = a_{11}$, $u_{12} = a_{12}$

$l_{21}u_{11} = a_{21} \implies l_{21} = \frac{a_{21}}{u_{11}}$. $l_{21}u_{12} + u_{22} = a_{22}$

$$u_{22} = a_{22} - u_{12}l_{21}$$

$u_{23} = a_{23}$, $l_{32}u_{22} = a_{32}$

$$\implies l_{32} = \frac{a_{32}}{u_{22}}$$

$l_{32}u_{23} + u_{33} = a_{33} \implies u_{33} = a_{33} - l_{32}u_{23}$.

$u_{34} = a_{34}$

$a_{34} = u_{33}l_{43} \implies l_{43} = \frac{a_{34}}{u_{33}}$.

$a_{44} = l_{43}u_{34} + u_{44} \implies u_{44} = a_{44} - l_{43}u_{34}$.

$\longrightarrow$ Solve the system of equations $2x-y=0$, $-x+2y-z=0$, $-y+2z-u=0$

$-z+2u=1$

Sol: Given that $2x-y=0$

$$-x+2y-z=0$$
$$-y+2z-u=0$$
$$-z+2u=1$$

The matrix equation of the given system of equations is $AX=B$ —— ①

Where $A=\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}$ is the Tri-diagonal matrix.

$X=\begin{bmatrix}x\\y\\z\\u\end{bmatrix}$ and $B=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

Now we solve this system by L-U decomposition method or Method of factorization.

Let $A=LU$ —— ②

Where $L=\begin{bmatrix}1&0&0&0\\l_{21}&1&0&0\\0&l_{32}&1&0\\0&0&l_{43}&1\end{bmatrix}$ and $U=\begin{bmatrix}u_{11}&u_{12}&0&0\\0&u_{22}&u_{23}&0\\0&0&u_{33}&u_{34}\\0&0&0&u_{44}\end{bmatrix}$

From ① and ②, $LUX=B$ —— ③

Taking $UX=Y$ —— ④ Where $Y=\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}$

From ③ and ④, $LY=B$.

$$LU=A \Longrightarrow \begin{bmatrix}1&0&0&0\\l_{21}&1&0&0\\0&l_{32}&1&0\\0&0&l_{43}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&0&0\\0&u_{22}&u_{23}&0\\0&0&u_{33}&u_{34}\\0&0&0&u_{44}\end{bmatrix}=\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}u_{11}&u_{12}&0&0\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&u_{23}&0\\0&l_{32}u_{22}&l_{32}u_{23}+u_{33}&u_{34}\\0&0&l_{43}u_{33}&l_{43}u_{34}+u_{44}\end{bmatrix}=\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}$$

Equating the corresponding elements on both sides, we get

$u_{11} = 2, \quad u_{12} = -1, \quad u_{23} = -1, \quad u_{34} = -1.$

$l_{21} u_{11} = -1 \implies l_{21} = \frac{-1}{u_{11}} = \frac{-1}{2}.$

$l_{21} u_{12} + u_{22} = 2 \implies u_{22} = 2 - l_{21} u_{12} = 2 - \left(-\frac{1}{2}\right)(-1) = \frac{3}{2}$

$l_{32} u_{22} = -1 \implies l_{32} = \frac{-1}{u_{22}} = -\frac{2}{3}.$

$l_{32} u_{23} + u_{33} = 2 \implies u_{33} = 2 - \left(-\frac{2}{3}\right)(-1) = \frac{4}{3}.$

$l_{43} u_{33} = -1 \implies l_{43} = \frac{-1}{u_{33}} = \frac{-3}{4}.$

$l_{43} u_{34} + u_{44} = 2 \implies u_{44} = 2 - \left(\frac{-3}{4}\right)(-1) = \frac{5}{4}.$

$$\therefore \; L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

From (5), first we have to find the values of y_1, y_2, y_3 and y_4.

$$\text{i.e } LY = B \implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ +1 \end{bmatrix}$$

Solving the system by forward substitution, we have $y_1 = 0.$

$$-\frac{1}{2} y_1 + y_2 = 0 \implies y_2 = 0$$

$$-2/3\, y_2 + y_3 = 0 \implies y_3 = 0.$$

$$-\frac{3}{4} y_3 + y_4 = 1 \implies y_4 = 1$$

$$\therefore \; Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Now from equation ④, we have to find the value of x, y, z and u.

$$UX = Y \Rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving the system by backward substitution, we have.

$$\frac{5}{4}u = 1 \Rightarrow u = \frac{4}{5}.$$

$$\frac{4}{3}z - u = 0 \Rightarrow \frac{4}{3}z = \frac{4}{5} \Rightarrow z = \frac{3}{5}.$$

$$\frac{3}{2}y - z = 0 \Rightarrow \frac{3}{2}y = z \Rightarrow \frac{3}{2}y = \frac{3}{5} \Rightarrow y = \frac{2}{5}.$$

$$2x - y = 0 \Rightarrow x = \frac{y}{2} = \frac{1}{5}.$$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ 3/5 \\ 4/5 \end{bmatrix}$$ is the solution of the given system.

→ Solve the system of equations $2x_1 + x_2 = 2$, $x_1 + 2x_2 + x_3 = 2$, $x_2 + 2x_3 + x_4 = 2$, $x_3 + 2x_4 = 1$.

Sol:- Given that

$$2x_1 + x_2 = 2$$
$$x_1 + 2x_2 + x_3 = 2$$
$$x_2 + 2x_3 + x_4 = 2$$
$$x_3 + 2x_4 = 1.$$

The matrix equation of the given system of equations is $AX = B$ —— ①.

Where $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ is the Tri diagonal matrix. $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

Now we solve this system by L-U decomposition method or Method of factorization.

Let $A = LU$ —— ② Where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}$ $U = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$

From ① & ②, We write $LUX = B$ —— ③

Taking $UX = Y$ —— ④ Where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

From ③ and ④, $LY = B$ —— ⑤

$$LU = A \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & u_{23} & 0 \\ 0 & l_{32}u_{22} & l_{32}u_{23}+u_{33} & u_{34} \\ 0 & 0 & l_{43}u_{33} & l_{43}u_{34}+u_{44} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Equating the corresponding elements on both sides, we get

$u_{11} = 2 \quad u_{12} = 1 \quad u_{23} = 1 \quad u_{34} = 1.$

$l_{21}u_{11} = 1 \Longrightarrow l_{21} = \frac{1}{u_{11}} = \frac{1}{2}$

$l_{21}u_{12} + u_{22} = 2 \Longrightarrow u_{22} = 2 - l_{21}u_{12} = 2 - \frac{1}{2}(1) = \frac{3}{2}$

$l_{32}u_{22} = 1 \Longrightarrow l_{32} = \frac{1}{u_{22}} = \frac{2}{3}$

$l_{32}u_{23} + u_{33} = 2 \Longrightarrow u_{33} = 2 - l_{32}u_{23} = 2 - \frac{2}{3}(1) = \frac{4}{3}.$

$l_{43}u_{33} = 1 \Longrightarrow l_{43} = \frac{1}{u_{33}} = \frac{3}{4}.$

$l_{43}u_{34} + u_{44} = 2 \Longrightarrow u_{44} = 2 - l_{43}u_{34} = 2 - \left(\frac{3}{4}\right)(1) = \frac{5}{4}.$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

Using equation ⑤, first we have to find the values of y_1, y_2, y_3 and y_4

$$LY = B \implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Solving the system by forward substitution, we have.

$y_1 = 2$

$\frac{1}{2} y_1 + y_2 = 2 \implies y_2 = 2 - \frac{1}{2}(2) = 1$

$\frac{2}{3} y_2 + y_3 = 2 \implies y_3 = 2 - \frac{2}{3} y_2 = 2 - \frac{2}{3}(1) = \frac{4}{3}$.

$\frac{3}{4} y_3 + y_4 = 1 \implies y_4 = 1 - \frac{3}{4} y_3 = 1 - \frac{3}{4}\left(\frac{4}{3}\right) = 0$.

$$\therefore \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4/3 \\ 0 \end{bmatrix}$$

Now using equation ④, we have to find the values of x_1, x_2, x_3 and x_4.

$$UX = Y \implies \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4/3 \\ 0 \end{bmatrix}$$

Solving the system by backward substitution, we have.

$\frac{5}{4} x_4 = 0 \implies x_4 = 0$.

$\frac{4}{3} x_3 + x_4 = \frac{4}{3} \implies x_3 = 1$

$\frac{3}{2} x_2 + x_3 = 1 \implies x_2 = 0$

$2x_1 + x_2 = 2 \implies x_1 = 1$

$$\therefore \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

<u>Note</u>:- In the method of decomposition or in the method of solving tri-diagonal system, we can take L and U such that L is unit lower triangular & U is upper triangular (or) L is lower triangular and U is unit upper triangular

LU - DECOMPOSITION METHOD

1) Solve the system $x+y+z=1$, $3x+y-3z=5$, $x-2y-5z=10$ by using the LU decomposition method. Ans:- $x=6$, $y=-7$, $z=2$.

2) Solve the system $4x+y+z=4$, $x+4y-2z=4$, $3x+2y-4z=6$ by using Method of factorization. Ans:- $x=1$, $y=\frac{1}{2}$, $z=\frac{-1}{2}$.

3) Solve the system $x_1+3x_2+8x_3=4$, $x_1+4x_2+3x_3=-2$, $x_1+3x_2+4x_3=1$ by Triangularisation Method. Ans:- $x_1=\frac{19}{4}$ $x_2=\frac{-9}{4}$, $x_3=\frac{3}{4}$.

4) Solve the following matrix equation by using the LU-decomposition method.

$$\begin{bmatrix} 3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

Ans:- $x=1$, $y=-2$, $z=1$

5) Solve the system of equations $x+y+z=3$, $x+2y+3z=6$, $x+y+4z=6$ by using Triangularisation Method. Ans:- $x=y=z=1$.

6) Solve the system of equations $10x+y+2z=13$, $3x+10y+z=14$, $2x+3y+10z=15$ by using Method of factorization Ans:- $x=y=z=1$.

7) Solve the system of equations $x+y-z=2$, $2x+3y+5z=-3$, $3x+2y-3z=6$ by using LU decomposition method. Ans:- $x=1$, $y=0$, $z=-1$.

8) Solve the system of equations $2x+y+4z=12$, $4x+11y-z=33$, $8x-3y+2z=20$ by using LU decomposition method Ans:- $x=3$, $y=2$, $z=1$.

9) Solve the following equations by expressing the coefficient matrix as a product of a lower triangular and upper triangular matrices.
$2x+y-z=3$, $x-2y-2z=1$, $x+2y-3z=9$ Ans:- $x=-\frac{1}{5}$, $y=\frac{7}{5}$, $z=-2$

10 Solve the following equations using LU decomposition method.
$10x_1+7x_2+8x_3+7x_4=32$, $7x_1+5x_2+6x_3+5x_4=23$, $8x_1+6x_2+10x_3+9x_4=33$
$7x_1+5x_2+9x_3+10x_4=31$. Ans:- $x_1=x_2=x_3=x_4=1$.

SOLUTION OF TRIDIAGONAL SYSTEMS.

1 Solve the following tridiagonal system of equations. $x_1 + 2x_2 = 7$, $x_1 - 3x_2 - x_3 = 4$, $4x_2 + 3x_3 = 5$. Ans: $x_1 = \frac{69}{11}$ $x_2 = \frac{4}{11}$, $x_3 = \frac{13}{11}$.

2 Solve the tridiagonal system of equations. $2x_1 - x_2 = 0$, $x_1 - 2x_2 + x_3 = 0$. $x_2 - 2x_3 + x_4 = 0$, $x_3 - 2x_4 = -1$. Ans:- $x_1 = \frac{1}{5}$ $x_2 = \frac{2}{5}$ $x_3 = \frac{3}{5}$ $x_4 = \frac{4}{5}$.

3 Solve the tridiagonal system of equations $2x - 3y = 8$, $3x + y + z = 4$, $y - 3z = -11$. Ans:- $x = 1$, $y = -2$, $z = 3$.

4 Solve the tridiagonal system $2x_1 - 3x_2 = 5$, $x_1 + 2x_2 - 3x_3 = -1$, $3x_2 - x_3 + 2x_4 = 1$ $x_3 + x_4 = 2$ Ans:- $x_1 = 1$ $x_2 = -1$, $x_3 = 0$, $x_4 = 2$

5 Solve the tridiagonal system $5x + 2y = 3$, $2x - 3y + z = 5$, $4y - 3z = -4$ Ans:- $x = 1$, $y = -1$, $z = 0$.

6 Solve the tridiagonal system $3x_1 + 2x_2 = 1$, $x_1 - 2x_2 + 3x_3 = -2$, $2x_2 - x_3 + x_4 = 1$ and $3x_3 - 4x_4 = 11$ Ans:- $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = -2$

Gaussian Elimination Method :—

This method of solving a system of n linear equations in n unknowns consists of eliminating the coefficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution.

Consider the system of non homogeneous equations.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \text{——— (1)}$$

The matrix equation of the given system of eqn's is $AX=B$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The augmented matrix of this system is.

$$[A|B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$R_2 \to R_2 - \frac{a_{21}}{a_{11}}R_1 \quad R_3 \to R_3 - \frac{a_{31}}{a_{11}}R_1$, we get

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

where $a'_{22} = a_{22} - \frac{a_{21}}{a_{11}}a_{12} \qquad a'_{23} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$

$a'_{32} = a_{32} - \frac{a_{31}}{a_{11}}a_{12} \qquad a'_{33} = a_{33} - \frac{a_{31}}{a_{11}}a_{13}$.

$b'_2 = b_2 - \frac{a_{21}}{a_{11}}b_1 \qquad b'_3 = b_3 - \frac{a_{31}}{a_{11}}b_1$

Here we assume that $a_{11} \neq 0$

We call $-\frac{a_{21}}{a_{11}}$, $-\frac{a_{31}}{a_{11}}$ as multipliers for the first stage.

a_{11} is called first pivot.

$$R_3 \longrightarrow R_3 - \frac{a'_{32}}{a'_{22}} R_2 \text{, weget}$$

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right] \quad \text{———} \quad (2)$$

$$\text{Where } a''_{33} = a'_{33} - \frac{a'_{32}}{a'_{22}} a'_{23}$$

$$b''_3 = b'_3 - \frac{a'_{32}}{a'_{22}} b'_2$$

We assume that $a'_{22} \neq 0$.

Here the multiplier is $-\frac{a'_{32}}{a'_{22}}$

New pivot is a'_{22}

The augmented matrix (4) corresponds to an upper triangular system which can be solved by backward substitution.

Note :-

(1) If one of the elements a_{11}, a'_{22}, a''_{33} are zero, the method is modified by rearranging the rows so that the pivot is non zero.

(2) This procedure is called partial pivoting.

(3) If this is impossible then the matrix is singular and the system has no solution.

Solve the equations $2x_1+x_2+x_3=10$, $3x_1+2x_2+3x_3=18$, $x_1+4x_2+9x_3=16$ using Gauss Elimination method.

Sol:- Given that
$$\left.\begin{aligned}2x_1+x_2+x_3&=10\\3x_1+2x_2+3x_3&=18\\x_1+4x_2+9x_3&=16\end{aligned}\right\} \longrightarrow (1)$$

The matrix equation of the given system of eqn's is $AX=B$

where $A=\begin{bmatrix}2&1&1\\3&2&3\\1&4&9\end{bmatrix}$ $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ $B=\begin{bmatrix}10\\18\\16\end{bmatrix}$

The augmented matrix of the given system is
$$[A|B]=\left[\begin{array}{ccc|c}2&1&1&10\\3&2&3&18\\1&4&9&16\end{array}\right]$$

$R_2\longrightarrow R_2-\frac{3}{2}R_1$ $\quad R_3\longrightarrow R_3-\frac{1}{2}R_1$
$$\sim\left[\begin{array}{ccc|c}2&1&1&10\\0&\frac{1}{2}&\frac{3}{2}&3\\0&\frac{7}{2}&\frac{17}{2}&11\end{array}\right]$$

$R_3\longrightarrow R_3-\dfrac{\frac{7}{2}}{\frac{1}{2}}R_2$ i.e $R_3\longrightarrow R_3-7R_2$
$$\sim\left[\begin{array}{ccc|c}2&1&1&10\\0&\frac{1}{2}&\frac{3}{2}&3\\0&0&-\frac{4}{2}&-10\end{array}\right]\sim\left[\begin{array}{ccc|c}2&1&1&10\\0&\frac{1}{2}&\frac{3}{2}&3\\0&0&-2&-10\end{array}\right]$$

The equivalent matrix equation of the given system of equations is
$$\begin{bmatrix}2&1&1\\0&\frac{1}{2}&\frac{3}{2}\\0&0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}10\\3\\-10\end{bmatrix}$$

The linear equations are

$$2x_1 + x_2 + x_3 = 10$$

$$\frac{x_2}{2} + \frac{3x_3}{2} = 3 \quad \text{i.e} \quad x_2 + 3x_3 = 6$$

$$-2x_3 = -10 \quad \text{i.e} \quad x_3 = 5$$

These equations can be solved by back substitution

$$x_2 = 6 - 3x_3$$

$$x_2 = 6 - 15 = -9$$

$$2x_1 = 10 - x_2 - x_3$$

$$2x_1 = 10 + 9 - 5 = 14$$

$$x_1 = 7.$$

$\therefore$ The solution of the given system is

$$x_1 = 7 \quad x_2 = -9 \quad x_3 = 5.$$

Gauss Jordan Method :—

This is modified Gauss Elimination method.

Consider the given system of linear equations in matrix form $AX = B$

Now reduce the augmented matrix $[A|B]$ by applying E-row operations only such that the coefficient matrix A is in diagonal form $[D|B']$ Then the solution is obtained directly.

(1) Using Gauss Jordan Method, Solve the system.

$2x+y+z=10$, $3x+2y+3z=18$, $x+4y+9z=16$.

Sol:- G/T $2x+y+z=10$ $3x+2y+3z=18$ $x+4y+9z=16$.

The matrix equation of the given system of equations is $AX=B$.

where $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

The augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$

$R_2 \to 2R_2 - 3R_1$ $R_3 \to 2R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{array}\right]$$

$R_3 \to R_3 - 7R_1$ $R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & -2 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$R_1 \to 2R_1 - R_3$ $R_2 \to 4R_2 + 3R_3$

$$\sim \left[\begin{array}{ccc|c} 4 & 0 & 0 & 28 \\ 0 & 4 & 0 & -36 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

This is of the form $[D|B']$.

The equivalent matrix equation of $AX=B$ is.

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ -36 \\ -20 \end{bmatrix}$$

$$4x = 28 \Rightarrow x = 7$$

$$4y = -36 \Rightarrow y = -9$$

$$-4z = -20 \Rightarrow z = 5.$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \\ 5 \end{bmatrix} \text{ is the solution.}$$

(2) Solve the system of equations by Gauss Jordan Method.

(a) $10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7.$

Ans:- $x = y = z = 1.$

(b) $10x_1 + x_2 + x_3 = 12$

$x_1 + 10x_2 - x_3 = 10$

$x_1 - 2x_2 + 10x_3 = 9.$

Ans:- $x_1 = x_2 = x_3 = 1.$

GAUSS ELIMINATION METHOD

1 Apply Gauss elimination method solve the equations $x+4y-z=-5$, $x+y-6z=-12$, $3x-y-z=4$.

Ans:- $x=1.6479$, $y=-1.1408$, $z=2.0845$

2 Solve $10x-7y+3z+5u=6$, $-6x+8y-z-4u=5$, $3x+y+4z+11u=2$, $5x-9y-2z+4u=7$ by Gauss elimination method.

Ans:- $x=5, y=4, z=-7, u=1$.

3 solve the following equations by Gauss elimination method.

$2x+y+z=10$, $3x+2y+3z=18$, $x+4y+9z=16$.

Ans:- $x=7$, $y=-9$ $z=5$

4 Solve $2x-y+3z=9$, $x+y+z=6$, $x-y+z=2$ by Gauss elimination method Ans:- $x=2$, $y=2$, $z=3$

5 Solve $2x_1+4x_2+x_3=3$, $3x_1+2x_2-2x_3=-2$, $x_1-x_2+x_3=6$ by Gauss elimination method. Ans:- $x_1=2$, $x_2=-1$, $x_3=3$

6 Solve $5x_1+x_2+x_3+x_4=4$, $x_1+7x_2+x_3+x_4=12$, $x_1+x_2+6x_3+x_4=-5$ $x_1+x_2+x_3+4x_4=-6$ Ans:- $x_1=1$, $x_2=2$, $x_3=-1$, $x_4=-2$

7 Solve (if possible) $2x+z=3$, $x-y+z=1$, $4x-2y+3z=3$

Ans:- In consistent.

8 Solve $4x-3y-9z+6w=0$, $2x+3y+3z+6w=6$, $4x-21y-39z-6w=-24$.

Ans:- $x=1+k_1-2k_2$, $y=(4-5k_1-2k_2)/3$ $z=k_1, w=k_2$.

9 Solve $2x_1+x_2+2x_3+x_4=6$, $6x_1-6x_2+6x_3+12x_4=36$, $4x_1+3x_2+3x_3-3x_4=-1$ $2x_1+2x_2-x_3+x_4=10$. Ans:- $x_1=2$, $x_2=1$ $x_3=-1$, $x_4=3$.

10. Solve $2x+3y-z=5$, $4x+4y-3z=3$, $2x-3y+2z=2$.

Ans:- $x=1, y=2, z=3$.

GAUSS JORDAN METHOD

1 Apply Gauss Jordan method, solve the equations $x+y+z=9$, $2x-3y+4z=13$, $3x+4y+5z=40$. Ans:- $x=1$ $y=3$ $z=5$

2. Solve by Gauss Jordan method $2x+5y+7z=52$, $2x+y-z=0$, $x+y+z=9$

Ans:- $x=1$, $y=3$, $z=5$

3 Solve by Gauss Jordan method $2x-3y+z=-1$, $x+4y+5z=25$, $3x-4y+z=2$ Ans:

4 Solve $x+3y+3z=16$, $x+4y+3z=18$, $x+4\cancel{z}+3y=19$ using Gauss Jordan method. Ans:- $x=1$ $y=2$, $z=3$.

5 Solve $2x+y+z=10$, $3x+2y+3z=18$, $x+4y+9z=16$ using Gauss Jordan method. Ans: $x=7$, $y=-9$, $z=5$

6 Apply Gauss Jordan method solve $2x_1+x_2+5x_3+x_4=5$, $x_1+x_2-3x_3+4x_4=-1$

$3x_1+6x_2-2x_3+x_4=8$, $2x_1+2x_2+2x_3-3x_4=2$

Ans:- $x_1=2$ $x_2=\frac{1}{5}$ $x_3=0$ $x_4=\frac{4}{5}$.

7 Solve $5x_1+x_2+x_3+x_4=4$, $x_1+7x_2+x_3+x_4=12$, $x_1+x_2+6x_3+x_4=-5$

$x_1+x_2+x_3+4x_4=-5$ by Gauss Jordan Method.

Ans:- $x_1=1$ $x_2=2$ $x_3=-1$ $x_4=-2$.

8 Solve $2x_1+x_2+2x_3+x_4=6$, $6x_1-6x_2+6x_3+12x_4=36$.

$4x_1+3x_2+3x_3-3x_4=-1$, $2x_1+2x_2-x_3+x_4=10$.

Ans:- $x_1=2$, $x_2=1$, $x_3=-1$, $x_4=3$.

R. NO	Q. NO
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
46 — 70	[illegible]

Vector :—

An ordered n-tuple of numbers is called an n-vector. The n numbers which are called components of the vector may be written in a horizontal or in a vertical line.

A vector over a real number is called a Real vector and vector over complex numbers is called a complex vector.

Eg: $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $[1\ 0\ 7]$ are two vectors.

Linearly dependent set of vectors :—

A set $\{x_1, x_2, x_3, \dots x_r\}$ of r vectors is said to be a linearly dependent set if there exist r scalars $k_1, k_2, k_3 \dots k_r$ not all zero such that $k_1 x_1 + k_2 x_2 + k_3 x_3 + \dots + k_r x_r = 0$ where 0 denotes the n vector with components all zero.

Linearly independent set of vectors :—

A set $\{x_1, x_2, x_3, \dots x_r\}$ of r vectors is said to be linearly independent set if the set is not linearly dependent i.e if $k_1 x_1 + k_2 x_2 + k_3 x_3 + \dots + k_r x_r = 0$. Where 0 denotes the n vector with components all zero.

(1) show that the system of vectors $(1,3,2)$ $(1,-7,-8)$ $(2,1,-1)$ linearly independent.

sol:- Let $a,b,c \in R$ then

$$a(1,3,2) + b(1,-7,-8) + c(2,1,-1) = \bar{0}$$

$$(a+b+2c,\ 3a-7b+c,\ 2a-8b-c) = (0,0,0)$$

$$a+b+2c=0 \quad 3a-7b+c=0 \quad 2a-8b-c=0.$$

$$a=3 \quad b=1 \quad c=-2.$$

$\therefore$ The given vectors are Linearly dependent.

(2) show that the system of vectors $(1,2,0)$ $(0,3,1)$ $(-1,0,1)$ is Linearly independent.

sol:- Let $a,b,c \in R$ then

$$a(1,2,0) + b(0,3,1) + c(-1,0,1) = \bar{0}$$

$$(a-c,\ 2a+3b,\ b+c) = (0,0,0)$$

$$a-c=0 \quad 2a+3b=0 \quad b+c=0.$$

$$a=0 \quad b=0 \quad b=0.$$

$\therefore$ The given vectors are Linearly independent.

Note:-

(i) If a set of vectors is linearly dependent then atleast one vector of the set can be expressed as a linear combination of the remaining vectors.

(ii) If a set of vectors is linearly independent then no vector of the set can be expressed as a linear combination of the remaining vectors.

System of Homogeneous Linear Equations :—

A set of equations of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = 0 \end{array}\right\} \text{—} ①$$

is said to be a system of m homogeneous equations in n unknowns $x_1, x_2, x_3 \dots x_n$.

The matrix form of the given system of equations ① is $AX = 0$ — ②

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m\times n}$ is called the coefficient matrix of the system of equations ①.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ is the matrix of unknowns and $O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$ is the null matrix of the system of equations ①.

Note :-

(1) The solution $x_1 = x_2 = x_3 = \dots = x_n = 0$ i.e $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = 0$ is called a solution of ① and it is said to be trivial solution (zero solution) of the system of homogeneous equations $AX = 0$.

(2) Always the system of homogeneous equations consistent i.e it contains a solution.

(3) Always a system of homogeneous linear equations i.e $AX = 0$ contains $n-r$ linearly independent solutions where r being the

rank of the co efficient matrix A and n being the number of variables of the system.

(4) The trivial solution $x=0$ is not linearly independent and it is a linearly dependent solution.

Nature of solutions of $AX=0$:-

Suppose we have m equations in n unknowns Then the coefficient matrix A will be of order $m \times n$. Let r be the rank of the matrix A.

case (i) :- If $r=n$, then the given system of equations $AX=0$ will have $n-r=n-n=0$ linearly independent solutions. So in this case the given system possesses a linearly dependent solution i.e only a trivial solution (zero solution).

case (ii) :- If $r<n$, then the given system of equations $AX=0$ $n-r$ linearly independent solutions. Any linear combination of these solutions will also be a solution of $AX=0$. Thus in this case the given system $AX=0$ contains an infinite number of solutions.

Case (iii) :— Suppose $m<n$ i.e the number of equations less than the number of unknowns. Since $r \leq m$, therefore r is definitely less than n.

Hence in this case the given system of equations must possess a non zero solution. So that the number of solutions of the system $AX=0$ will be infinite.

Working Rule :—

Step 1 :- First write the matrix equation of the given System of equations.

Step 2 :— Reduce the co efficient matrix A to echelon form to determine the rank of A. Let r be the rank of the coefficient matrix A of order $m \times n$ and n be the number of variables or unknowns of the given system of eqn $AX = 0$.

Step 3 :— Case (i) :- If $r = n$, then the given system of equations $AX = 0$ possesses only a trivial trivial solution (zero sol.) i.e $x_1 = 0$ $x_2 = 0$ . . . $x_n = 0$ or $X = 0$.

Case (ii) :- If $r < n$, then the given System of equations possesses an infinite number of solutions. Of these solutions, $(n-r)$ solutions are linearly independent and the remaining are depending upon them. So we have to assign arbitrary values for $(n-r)$ variables and the remaining variables are depending upon them.

Case (iii) :- If $m < n$, then since $r \le m < n$, here also the given system possesses an infinite number of solutions.

Note :- (1) If A is a non singular matrix i.e $|A| \neq 0$ then the linear System $AX = 0$ has only a trivial solution (zero solution).

(2) If A is a singular matrix i.e $|A| = 0$, then the linear System $AX = 0$ contains a non zero solution i.e we get an infinite number of Solutions.

(1) solve completely the system of equations.

$x+y-3z+2w=0$, $2x-y+2z-3w=0$, $3x-2y+z-4w=0$.

$-4x+y-3z+w=0$.

Sol:- Given that $x+y-3z+2w=0$

$2x-y+2z-3w=0$

$3x-2y+z-4w=0$

$-4x+y-3z+w=0$ $\longrightarrow$ There are 3 eqn's 3 unknowns x, y and z.

The matrix equation of the given system of equations is $AX=0$.

where $A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}$ is the coefficient matrix of the given system of equations and $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}$$

Now we have to reduce the coefficient matrix A to echelon form by applying E-row transformations only and determine the rank of A.

$R_2 \longrightarrow R_2-2R_1$, $R_3 \longrightarrow R_3-3R_1$ $R_4 \longrightarrow R_4+4R_1$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

$R_3 \longrightarrow 3R_3-5R_2$, $R_4 \longrightarrow 3R_4+5R_2$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & -5 & -8 \end{bmatrix}$$

$R_4 \to 2R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 0 & -21 \end{bmatrix}$$

$\therefore$ $\rho(A) = r = 4 =$ The No. of non zero rows of equivalent to matrix A.

i.e $r = 4 = n$ i.e. the number of unknowns of the given system.

Hence the given system of equations contains only a trivial solution

$\therefore$ $x = y = z = w = 0$ is the only solution of the given system of equations.

→ Solve completely the system of equations

$x - 2y + z - w = 0$ $\quad x + y - 2z + 3w = 0$ $\quad 4x + y - 5z + 8w = 0$ and

$5x - 7y + 2z - w = 0$

sol:- Given that $x - 2y + z - w = 0$

$x + y - 2z + 3w = 0$

$4x + y - 5z + 8w = 0$

$5x - 7y + 2z - w = 0$.

→ These are 3 eqns in 4 unknowns x, y and z, w.

The matrix form of the given system of equations is $AX = O$.

Where $A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$ $\quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ $\quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

Now we reduce the matrix A to echelon form by applying E-row operations only

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - 4R_1 \quad R_4 \to R_4 - 5R_1$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2 \,,\ R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form.

Here $\rho(A) = r = 2 =$ The No. of non zero rows equivalent to A.

$\rho(A) = 2 < 4$ (No. of unknowns)

So that the given system possesses an infinite no. of sol's. and the remaining of these $n - r = 4 - 2 = 2$ are linearly independent and the remaining are depending upon them.

So we have to assign arbitrary values for 2 variables and the remaining 2 variables are depending upon them.

Now the equivalent matrix eqn of $AX = 0$ is

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear eqn's are

$$x - 2y + z - w = 0$$

$$3y - 3z + 4w = 0.$$

choose $y = K_1 \quad z = K_2$

$$4w = 3z - 3y$$

$$w = \frac{3K_2 - 3K_1}{4}$$

$$x = 2y - z + w$$

$$= 2K_1 - K_2 + \frac{3K_2 - 3K_1}{4}$$

$$x = \frac{5K_1 - K_2}{4}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \frac{5K_1 - K_2}{4} \\ K_1 \\ K_2 \\ \frac{3K_2 - 3K_1}{4} \end{bmatrix} = K_1 \begin{bmatrix} \frac{5}{4} \\ 1 \\ 0 \\ \frac{-3}{4} \end{bmatrix} + K_2 \begin{bmatrix} \frac{-1}{4} \\ 0 \\ 1 \\ \frac{3}{4} \end{bmatrix}$$ is the general solution of the given system of equations.

→ Solve completely the system of equations
$4x + 2y + z + 3u = 0$
$6x + 3y + 4z + 7u = 0$
$2x + y + u = 0$.

Sol:- Given that
$$4x + 2y + z + 3u = 0$$
$$6x + 3y + 4z + 7u = 0$$
$$2x + y + u = 0.$$

→ There are 3 eqn's in 4 unknowns x, y, z and u.

The matrix form of the given system of equations is $AX = O$

where $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Now we reduce the matrix A to echelon form by applying E-row operations only.

$R_2 \longrightarrow 2R_2 - 3R_1 \quad R_3 \longrightarrow 2R_3 - R_1$

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$R_3 \longrightarrow 5R_3 + R_2$

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form.

Here $\rho(A) = 2 = r =$ The No. of non zero rows equivalent to A.

$\rho(A) = 2 < 4$ (No. of unknowns)

So that the given system of equations has an infinite no. of solutions. Of these solutions, $n - r = 4 - 2 = 2$ are linearly independent and the remaining are depending upon them.

So we have to assign arbitrary values for 2 variables and the remaining 2 variables are depending upon them.

Now the equivalent matrix equation of $AX = 0$ is

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear equations are

$$4x + 2y + z + 3u = 0$$

$$z + u = 0$$

$$y = k_1$$

$$z = k_2$$

$$u = -z$$

$$u = -K_2$$

$$4x = -2y - z - 3u.$$

$$= -2K_1 - K_2 + 3K_2$$

$$4x = 2K_2 - 2K_1$$

$$x = \frac{K_2 - K_1}{2}$$

$$\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} \frac{K_2 - K_1}{2} \\ K_1 \\ K_2 \\ -K_2 \end{bmatrix} = K_1 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ -1 \end{bmatrix}$$ is the general solution

of given system of equations where K_1, K_2 are arbitrary constants.

Here the two L.I solutions are $X_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

→ Solve the following system of equations for all values of k.

$2x+3ky+(3k+4)z=0$, $x+(k+4)y+(4k+2)z=0$,

$x+2(k+1)y+(3k+4)z=0$.

Sol:- Given that the system of equations are

$$\left.\begin{array}{l} 2x+3ky+(3k+4)z=0 \\ x+(k+4)y+(4k+2)z=0 \\ x+2(k+1)y+(3k+4)z=0 \end{array}\right] \longrightarrow (1)$$

→ There are 3 eqn's in 3 unknowns x, y and z.

The matrix form of the given system of equations is $AX=O$ —— (2)

$$\text{Where} \quad A=\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2(k+1) & 3k+4 \end{bmatrix} \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad O=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We know that If the coefficient matrix A is singular i.e $|A|=0$ then the linear system $AX=0$ contains a non zero solution i.e we get an infinite number of solution.

$$|A|=0 \quad \text{i.e} \quad \begin{vmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{vmatrix}=0.$$

$R_1 \leftrightarrow R_2$

$$\begin{vmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{vmatrix}=0$$

$R_2 \to R_2-2R_1 \qquad R_3 \to R_3-R_1$

$$\begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -(k+2) \end{vmatrix}=0$$

$$(k-2)\begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & 1 & -1 \end{vmatrix}=0.$$

$$(K-2)\left[\cancel{8}-K+5K\right]=0$$

$$(K-2)(4K+8)=0$$

$$K=\pm 2.$$

Case (i):- When $K \neq \pm 2$, then the given system of equations possesses a zero solution i.e trivial solution. i.e $x=y=z=0$.

Case (ii):- When $K=2$

$$A=\begin{bmatrix} 2 & 6 & 10 \\ 1 & 6 & 10 \\ 1 & 6 & 10 \end{bmatrix}$$

Now reduce the matrix A to echelon form by applying elementary row operations and determine the rank of A.

$$R_2 \to 2R_2 - R_1, \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 2 & 6 & 10 \\ 0 & 6 & 10 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{Which is in echelon form.}$$

$\therefore$ $\rho(A)=2$ = The No. of non zero rows equivalent to A.

$\therefore$ $\rho(A)=2<3$ (No. of unknowns)

So that the given system of eqns contains an infinite number of solutions. of these $n-r=3-2=1$ L.I solutions.

We have to assign an arbitrary values for 1 variable. and remaining 2 variables are depending upon them.

The equivalent matrix equation of $AX=0$ is

$$\begin{bmatrix} 2 & 6 & 10 \\ 0 & 6 & 10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear equations are

$$2x + 6y + 10z = 0$$
$$6y + 10z = 0$$

choose $y = K_1$

$$10z = -6y$$
$$z = -\frac{3}{5}K_1$$

$$x = -3y - 5z$$
$$x = -3K_1 - 5.\left(-\frac{3}{5}\right)K_1$$
$$x = 0.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ K_1 \\ -\frac{3}{5}K_1 \end{bmatrix}$$ where K_1 is an arbitrary constant.

Case (iii):- When $K = -2$

$$A = \begin{bmatrix} 2 & -6 & -2 \\ 1 & 2 & -6 \\ 1 & -2 & -2 \end{bmatrix}$$

Now reduce the matrix A to echelon form by applying E-row operations only and determine the rank of A.

$R_2 \to 2R_2 - R_1$, $R_3 \to 2R_3 - R_1$

$$\sim \begin{bmatrix} 2 & -6 & -2 \\ 0 & 10 & -10 \\ 0 & 2 & -2 \end{bmatrix}$$

$R_3 \to 5R_3 + R_2$

$$\sim \begin{bmatrix} 2 & -6 & -2 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form.

$\therefore \rho(A) = 2 =$ The no. of non zero rows equivalent to A.

$\therefore$ $\rho(A) = 2 < 3$ (No. of unknowns)

Sothat the given system of eqn's contains an infinite no. of solutions of these $n-r = 3-2 = 1$ L.I solution.

We have to assign an arbitrary values for $n-r = 3-2 = 1$ variable.

and remaining 2 variables are depending upon them.

The equivalent matrix equation of $AX = 0$ is

$$\begin{bmatrix} 2 & -6 & -2 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear eqn's are $2x - 6y - 2z = 0$

$10y - 10z = 0.$

$\Rightarrow y - z = 0.$

choose $y = k_1$

$z = y = k_1.$

$$x = \frac{6y + 2z}{2} = \frac{6k_1 + 2k_1}{2}$$

$x = 4k_1.$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$ is the sol. of given system when $k = -2$, where k_1 is an arbitrary constant

→ Find the values of k for which the equations $(k-1)x + (3k+1)y + 2kz = 0$, $(k-1)x + (4k-2)y + (k+3)z = 0$, $2x + (3k+1)y + (3k-3)z = 0$ are consistent and find the ratios of $x:y:z$ when k has the smallest of these values. What happens when k has the greatest of these values.

Sol: Given that

$$(k-1)x + (3k+1)y + 2kz = 0$$
$$(k-1)x + (4k-2)y + (k+3)z = 0$$
$$2x + (3k+1)y + (3k-3)z = 0$$

→ These are 3 equ's in 3 unknowns x, y and z

The matrix form of the given system of equations is $AX = O$.

Where $A = \begin{bmatrix} k-1 & 3k+1 & 2k \\ k-1 & 4k-2 & k+3 \\ 2 & 3k+1 & 3k-3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We know that If the coefficient matrix A is singular i.e $|A| = 0$ then the linear system $AX = 0$ contains a non zero solution i.e we get an infinite number of solutions.

$|A| = 0$ i.e $\begin{vmatrix} k-1 & 3k+1 & 2k \\ k-1 & 4k-2 & k+3 \\ 2 & 3k+1 & 3k-3 \end{vmatrix} = 0$

$R_2 \to R_2 - R_1$

$$\begin{vmatrix} k-1 & 3k+1 & 2k \\ 0 & k-3 & 3-k \\ 2 & 3k+1 & 3k-3 \end{vmatrix} = 0$$

$C_3 \to C_3 + C_2$

$$\begin{vmatrix} k-1 & 3k+1 & 5k+1 \\ 0 & k-3 & 0 \\ 2 & 3k+1 & 6k-2 \end{vmatrix} = 0.$$

$$(k-3)\left[(k-1)(6k+2) - 2(5k+1)\right] = 0.$$

$$(k-3)\,6k\,(k-3) = 0$$

$$k\,(k-3)^2 = 0$$

$$\therefore \; k = 0, 3, 3.$$

Case (i) When $k=0$:-

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{bmatrix}$$

Now reduce the matrix A into Echelon form by applying elementary row operations only

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in Echolon form.

$\rho(A) = 2 =$ No. of Non zero rows equivalent to A.

$\rho(A) = 2 < 3$ (No. of unknowns)

So that the given system of equations contains an infinite no. of solutions. of these $n-r = 3-2 = 1$ L.I solution.

To determine this, we have to assign an arbitrary values for $n-r = 3-2 = 1$ variable.

An equivalent matrix equation of $AX=0$ is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear equations are

$-x+y=0$

$-3y+3z=0 \Rightarrow y-z=0.$

Choose $z = k_1$.

$y - z = 0 \implies y = z = k_1$

$-x + y = 0 \implies x = y = k_1$.

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ k_1 \end{bmatrix}$ is the solution of given system

$\therefore x : y : z = 1 : 1 : 1$

Case (ii) when $k = 3$:-

$$A = \begin{bmatrix} 2 & 10 & 6 \\ 2 & 10 & 6 \\ 2 & 10 & 6 \end{bmatrix}$$

When $k = 3$, The system of equations $AX = 0$ becomes identical.

→ Solve the system completely for all values of λ, $\lambda x + y + z = 0$, $x + \lambda y + z = 0$, $x + y + \lambda z = 0$.

Sol:- Given that $\lambda x + y + z = 0$, $x + \lambda y + z = 0$, $x + y + \lambda z = 0$ —— ①

→ There are 3 eqn's in 3 unknowns x, y and z.

The matrix form of the given system ① is $AX = 0$

where $A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

We know that If the coefficient matrix A is singular i.e $|A| = 0$ then the linear system $AX = 0$ contains a non zero solution i.e. we get an infinite no. of solutions .

$$|A| = 0 \quad \text{i.e} \quad \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_1 + \lambda_3$

$$\begin{vmatrix} \lambda+2 & \lambda+2 & \lambda+2 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$(\lambda+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$C_2 \rightarrow C_2 - C_1 \qquad C_3 \rightarrow C_3 - C_1$$

$$(\lambda+2)\begin{vmatrix} 1 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ 1 & 0 & \lambda-1 \end{vmatrix} = 0$$

$$(\lambda+2)(\lambda-1)^2 = 0$$

$$\lambda = -2, 1, 1.$$

Case(i):- When $\lambda \neq 1, -2$, then the given system of equations posseses a zero solution i.e trivial solution.

$$\therefore \ x = y = z = 0.$$

Case (ii):- When $\lambda = -2$

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Now reduce the matrix A to echelon form by applying elemen-tary row operations only and determine the rank of A.

$$R_2 \rightarrow 2R_2 + R_1, \quad R_3 \rightarrow 2R_3 + R_1$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2.$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \ \rho(A) = 2$ = The No. of non zero rows equivalent to A

$\rho(A) = 2 < 3$ (No. of unknowns)

Sothat the given system of eqn's contains an infinite no. of solutions. of these $n-r = 3-2 = 1$ L.I solution.

To determine this, we have to assign an arbitrary values for $n-r = 3-2 = 1$ variable and remaining 2 variables are depending upon them.

The equivalent matrix eqn. of $AX = O$ is

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear eqn's are $-2x+y+z=0$

$-3y+3z=0 \Rightarrow y-z=0$

choose $z = k_1$

$y = z = k_1$

$2x = y+z$

$2x = k_1+k_1$

$x = k_1$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ k_1 \end{bmatrix}$$ where k_1 is arbitrary constant, is the solution of the given system when $\lambda = -2$.

Case (iii) :- when $\lambda = 1$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Now reduce the matrix A to echelon form by applying elementary row operations only and determine the rank of A.

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_1$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore\ \rho(A) = 1 =$ The No. of non zero rows equivalent to A.

$\rho(A) = 1 < 3$ (No. of unknowns)

Sothat the given system of eqn's contains an infinite no. of solutions of these $n-r = 3-1 = 2$ L.I solutions.

To determine this, we have to assign an arbitrary values for $n-r = 3-1 = 2$ variables and remaining 1 variable. is depending upon them.

The equivalent matrix equation of $AX = 0$ is

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

The linear eqn. is $x+y+z=0$

choose $y = k_1,\ z = k_2$

$x = -y-z$

$x = -k_1 - k_2$

$$\therefore\ X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-k_1-k_2\\ k_1\\ k_2\end{bmatrix} = k_1\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix} + k_2\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$$

Where k_1 and k_2 are arbitrary constants, is the solution of the given system.

Here the two L.I solutions are $X_1 = \begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$ $X_2 = \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$.

→ Show that the only real number λ for which the system.

$x+2y+3z=\lambda x$, $3x+y+2z=\lambda y$, $2x+3y+z=\lambda z$ has a non zero solution is 6 and solve them when $\lambda=6$.

Sol:- Given system can be written as

$$\left.\begin{aligned}(1-\lambda)x+2y+3z&=0.\\ 3x+(1-\lambda)y+2z&=0\\ 2x+3y+(1-\lambda)z&=0\end{aligned}\right\} \longleftarrow (1)$$

These are 3 eqn's in 3 unknowns x, y and z.

The matrix form of the given system is $AX=O$.

Where $A=\begin{bmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ $O=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

We know that If the coeff. matrix A is singular i.e $|A|=0$. then the linear system $AX=0$ contains a non zero solution i.e we get an infinite no. of solutions.

$$|A|=0 \text{ i.e } \begin{vmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$R_1 \to R_1+R_2+R_3$

$$\begin{vmatrix}6-\lambda & 6-\lambda & 6-\lambda\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$$(6-\lambda)\begin{vmatrix}1 & 1 & 1\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$C_2 \to C_2-C_1,\ C_3 \to C_3-C_1$

$$(6-\lambda)\begin{vmatrix}1 & 0 & 0\\ 3 & -\lambda-2 & -1\\ 2 & 1 & -\lambda-1\end{vmatrix}=0$$

$$(6-\lambda)\left[(\lambda+2)(\lambda+1)+1\right]=0$$

$$(6-\lambda)(\lambda^2+3\lambda+3)=0$$

$$\lambda=6,\ -\frac{3}{2}\pm\frac{\sqrt{3}}{2}i$$

$\therefore$ The given system have non zero solution for only real number $\lambda=6$.

Case (i) When $\lambda=6$.

$$A=\begin{bmatrix}-5 & 2 & 3\\ 3 & -5 & 2\\ 2 & 3 & -5\end{bmatrix}$$

Now reduce the matrix A to echelon form by applying elementary row operations only and determine the rank of A.

$R_2 \to 5R_2+3R_1 \quad R_3 \to 2R_1+5R_3$

$$\sim\begin{bmatrix}-5 & 2 & 3\\ 0 & -19 & 19\\ 0 & 19 & -19\end{bmatrix}$$

$R_3 \to R_3+R_2$

$$\sim\begin{bmatrix}-5 & 2 & 3\\ 0 & -19 & 19\\ 0 & 0 & 0\end{bmatrix}$$

Which is in echelon form.

$\rho(A)=2=$ The no. of non zero rows equivalent to A.

$\rho(A)=2<3$ (No. of unknowns).

Sothat the given system of eqn's contains an infinite no. of solutions of these $n-r=3-2=1$ L.I solution.

To determine this we have to assign an arbitrary values for $n-r=3-2=1$ variable and remaining 2 variables are depending upon them.

The equivalent matrix eqn. of $AX=0$ is

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear eqn's are $-5x+2y+3z=0$

$$-19y+19z=0$$
$$\Rightarrow y-z=0$$

choose $z=k_1$

$$y=z=k_1$$
$$5x = 2y+3z$$
$$5x = 2k_1+3k_1$$
$$x = k_1$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ where k_1 is an arbitrary constant

is the solution of the given system.

SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS.

1 Find all the solutions of the following homogeneous systems.

(a) $3x+y+2z=0, \quad x-2y+3z=0, \quad x+5y-4z=0.$

Ans: $x=-k, \quad y=z=k.$

(b) $x+y+2z=0, \quad 3x+4y-7z=0, \quad -x-2y+11z=0.$

Ans: $x=-15k, y=13k, z=k.$

(c) $x+2y+3z+4w=0, \quad x+y+z+w=0, \quad x+2y+6z+12w=0.$

Ans:- $x=-2\alpha/3, \quad y=7\alpha/3 \quad z=-8\alpha/3 \quad w=\alpha.$

(d) $x+y+z+w=0, \quad -x+y+z-w=0, \quad -x-y+z+w=0, \quad x+y-z+w=0$

Ans:- $x=y=z=w=0.$

(e) $2x-y-3z+w=0, \quad x+y+z+w=0, \quad 2x-7y-13z-w=0, \quad -x+5y+9z+w=0.$

Ans: $x=\frac{2}{3}(k_2-k_1), \quad y=-\frac{1}{3}(5k_2+k_1) \quad z=k_2 \quad w=k_1$

(f) $3x+y+z+4w=0, \quad 4y+10z+w=0, \quad x+7y+17z+3w=0, \quad 2x+2y+4z+3w=0.$

Ans:- $x=(2\beta-5\alpha)/4 \quad y=-(10\beta+\alpha)/4 \quad z=\beta, w=\alpha.$

(g) $3x-11y+5z=0, \quad 4x+y-10z=0, \quad 4x+9y-6z=0$ Ans:- $x=y=z=0.$

(h) $x+y-3z+2w=0, \quad 2x-y-2z-3w=0, \quad 3x-5y-w=0, \quad 5x-y-7z-4w=0.$

Ans:- $x=(\alpha+5\beta)/3 \quad y=(4\beta-7\alpha)/3 \quad z=\beta \quad w=\alpha.$

(i) $x+y-2z-w=0, \quad 2x+y-z-2w=0, \quad 3x+2y-z-3w=0, \quad 4x+2y+2z-4w=0.$

Ans:- $x=k_1-2k_2, \quad y=5k_2, \quad z=k_2 \quad w=k_1.$

(j) $3x-11y+5z=0, \quad 4x+y-10z=0, \quad 4x+9y-6z=0$ Ans:- $x=y=z=0.$

2 If a, b, c are distinct non zero numbers show that the homogeneous system $\begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has no non trivial solution.

3 Solve the system $2x+y+2z=0, \quad x+y+3z=0, \quad 4x+3y+8z=0$

Ans:- $x=k \quad y=-4k \quad z=k.$

4 Determine the values of λ for which the following set of equations may possess non trivial solution. $3x_1 + x_2 - \lambda x_3 = 0$, $4x_1 - 2x_2 - 3x_3 = 0$. $2\lambda x_1 + 4x_2 + \lambda x_3 = 0$. Ans: $\lambda = 1, -9$; $x_1 = 2k$, $x_2 = 6k$, $x_3 = -\frac{4}{3}k$;

$x_1 = k_1$ $x_2 = -k_1$ $x_3 = 2k_1$.

5 Solve the system of equations $x + 2y + (2+k)z = 0$, $2x + (2+k)y + 4z = 0$. $7x + 13y + (18+k)z = 0$ for all values of k. Ans: $k = 1, \frac{4}{3}$.

$x = 1$ $y = -2k$ $z = k$; $x = \frac{14}{3}k$ $y = -4k$ $z = k$.

6 Solve the system $\lambda x + y + z = 0$, $x + \lambda y + z = 0$, $x + y + \lambda z = 0$ if the system has non zero solution only Ans: $\lambda = 1, -2$, $x = -k_1 - k_2$, $y = k_1$, $z = k_2$;

$x = y = z = k$.

7 Show that the only real number λ for which the system $x + 2y + 3z = \lambda x$, $3x + y + 2z = \lambda y$, $2x + 3y + z = \lambda z$ has non zero solution is 6 and solve them when $\lambda = 6$.

8 Solve $2x + 3ky + (3k+4)z = 0$, $x + (k+4)y + (4k+2)z = 0$, $x + 2(k+1)y + (3k+4)z = 0$

Ans: $k = 2, -2$; $x = 0$ $y = -\frac{5}{3}k_1$ $z = k_1$; $x = 4k_2$, $y = k_2$ $z = k_2$.

9 Find the values of λ for which the equations $(\lambda-1)x + (3\lambda+1)y + 2\lambda z = 0$. $(\lambda-1)x + (4\lambda-2)y + (\lambda+3)z = 0$, $2x + (3\lambda+1)y + 3(\lambda-1)z = 0$. are consistent and find the ratio of $x:y:z$ when λ has the smallest of these values. What happens λ has the greatest of these values.

10 Show that the system of equations $2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$ can possess a non trivial solution only if $\lambda = 1$, $\lambda = -3$ obtain the general solution in each case.

11 Solve $4x + 2y + z + 3u = 0$, $2x + y + u = 0$, $6x + 3y + 4z + 7u = 0$

Ans: $x = -\frac{1}{2}(c_1 + c_2)$ $y = c_1$, $z = -c_2$ and $u = c_2$.

12 Solve $x + 3y - 2z = 0$, $2x - y + 4z = 0$, $x - 11y + 14z = 0$.

Ans: $x = -\frac{10}{7}k$, $y = \frac{8}{7}k$ $z = k$.

Problems on L.I and L.D set of vectors :-

→ Examine the following vectors for linear dependence or independence. If dependent, find the relation amongest them.

$x_1 = (2,-1,3,2)$. $x_2 = (3,-5,2,2)$ $x_3 = (1,3,4,2)$

Sol:- Given that $x_1 = (2,-1,3,2)$ $x_2 = (3,-5,2,2)$ $x_3 = (1,3,4,2)$.

Let $K_1 x_1 + K_2 x_2 + K_3 x_3 = \bar{0}$

$K_1(2,-1,3,2) + K_2(3,-5,2,2) + K_3(1,3,4,2) = \bar{0}$.

$(2K_1 + 3K_2 + K_3,\ -K_1 - 5K_2 + 3K_3,\ 3K_1 + 2K_2 + 4K_3,\ 2K_1 + 2K_2 + 2K_3) = (0,0,0,0)$

Equating corresponding components,

$$\begin{aligned} 2K_1 + 3K_2 + K_3 &= 0. \\ -K_1 - 5K_2 + 3K_3 &= 0 \quad \text{—— ①} \\ 3K_1 + 2K_2 + 4K_3 &= 0 \\ 2K_1 + 2K_2 + 2K_3 &= 0. \end{aligned}$$

→ There are 4 eqn's in 3 unknowns.

The matrix form of the system ① is $AX = O$.

Where $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -5 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}$ $X = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -5 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}$$

Now reduce the matrix A into echelon form by applying elementary row operations only.

$R_2 \to 2R_2 + R_1,\ R_3 \to 2R_3 - 3R_1,\ R_4 \to R_4 - R_1$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -7 & 7 \\ 0 & -5 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_3 \to 7R_3 - 5R_2, \quad R_4 \to 7R_4 - R_2$

$$\sim \begin{bmatrix} 2 & 3 & 1 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form.

$\rho(A) = 2$ = No. of non zero rows of last equivalent to A.

$\therefore \rho(A) = 2 < 3$ (No. of unknown)

Sothat the given system have an infinite no. of solutions (Non trivial)

of these $n - r = 3 - 2 = 1$ L.I solution.

To determine this we have to assign an arbitrary value for $n-r = 3-2=1$

variable and the remaining are depending upon them.

Now the equivalent matrix equation of $AX = 0$ is.

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$2k_1 + 3k_2 + k_3 = 0$

$-7k_2 + 7k_3 = 0 \Rightarrow k_2 - k_3 = 0.$

choose $k_3 = t$

$k_2 = k_3 = t.$

$k_1 = \dfrac{-3k_2 - k_3}{2} = \dfrac{-3t - t}{2}$

$k_1 = -2t$

$\therefore k_1 = -2t \quad k_2 = t \quad k_3 = t$

Since $k_1\ k_2\ k_3$ are not all zero, the vectors are L.I.

We have $k_1 x_1 + k_2 x_2 + k_3 x_3 = 0.$

$-2t x_1 + t x_2 + t x_3 = 0.$

$2x_1 = x_2 + x_3.$

Examine for linear dependence or independence of vectors $x_1 = (1,1,-1)$ $x_2 = (2,3,5)$ $x_3 = (2,-1,4)$. If dependent find the relation between them.

Sol:- Given that $x_1 = (1,1,-1)$ $x_2 = (2,3,5)$ $x_3 = (2,-1,4)$

$$\text{Let } K_1 x_1 + K_2 x_2 + K_3 x_3 = \bar{0}$$

$$K_1(1,1,-1) + K_2(2,3,5) + K_3(2,-1,4) = \bar{0}$$

$$(K_1 + 2K_2 + 2K_3,\ K_1 + 3K_2 - K_3,\ -K_1 + 5K_2 + 4K_3) = (0,0,0)$$

Equating corresponding components.

$$K_1 + 2K_2 + 2K_3 = 0 \quad \text{——— (1)}$$
$$K_1 + 3K_2 - K_3 = 0$$
$$-K_1 + 5K_2 + 4K_3 = 0$$

There are 3 eqn's in 3 unknowns.

The matrix form of the given system (1) is $AX = O$.

where $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & -1 \\ -1 & 5 & 4 \end{bmatrix}$ $X = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & -1 \\ -1 & 5 & 4 \end{bmatrix}$$

Now reduce the matrix A into echelon form by applying elementary row operations only.

$$R_2 \to R_2 - R_1,\ R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 7 & 6 \end{bmatrix}$$

$$R_3 \to R_3 - 7R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 27 \end{bmatrix}$$

$\to$ which is in echelon form

$\rho(A) = 3 =$ No. of non zero rows of last equivalent to A.

$\therefore\ \rho(A) = 3 =$ No. of unknown.

So that the given system have trivial solution (zero solution)

$$K_1 = 0 \quad K_2 = 0 \quad K_3 = 0.$$

Since K_1, K_2, K_3 are all zero, the vectors are linearly independent.

Linearly independent and Linearly dependent set of vectors.

1) Examine for linear dependence the system of vectors $(1,2,-1,0)$, $(1,3,1,2)$ $(4,2,1,0)$, $(6,1,0,1)$ and if dependent, find the relation between them. Ans:- Linearly independent.

2) Examine whether following vectors are linearly independent or dependent $x_1 = (2,2,1)^T$ $x_2 = (1,3,1)^T$ $x_3 = (1,2,2)^T$

Ans:- Linearly independent.

3) Examine whether following vectors are linearly independent or dependent $x_1 = (3,1,1)$ $x_2 = (2,0,-1)$ $x_3 = (4,2,1)$

Ans:- Linearly independent.

4) Examine whether following vectors are linearly independent or dependent $x_1 = (1,1,-1)$ $x_2 = (2,3,-5)$ $x_3 = (2,-1,4)$

Ans:- Linearly independent

5) Examine for linear dependence or independence of the following vectors. If dependent, find the relation between them. $x_1 = (1,-1,1)$ $x_2 = (2,1,1)$ $x_3 = (3,0,2)$ Ans:- Linearly dependent, $x_1 + x_2 = x_3$.

6) Examine for linear dependence or independence of the following vectors. If dependent find the relation between them.

$x_1 = (1,1,1,3)$ $x_2 = (1,2,3,4)$ $x_3 = (2,3,4,7)$

Ans:- Linearly dependent, $x_1 + x_2 = x_3$.

7) Show that the vectors $x_1 = (1,-1,2,2)^T$ $x_2 = (2,-3,4,-1)^T$, $x_3 = (-1,2,-2,3)^T$ are linearly dependent. Hence find the relation b/n them Ans:- $x_1 = x_2 + x_3$

8) Show that the vectors $x_1 = (3,1,-4)'$ $x_2 = (2,2,-3)$ $x_3 = (0,-4,1)$ are linearly dependent Hence find the relation b/n them.

Ans:- $2x_1 = 3x_2 + x_3$.

Vector Space :-

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every u, v, w in V and scalars c and d, then V is called a vector space (over the reals $\mathbb{R}$)

(1) Addition.

(a) $u+v$ is a vector in V (closure under addition)

(b) $u+v = v+u$. (Commutative property of addition)

(c) $(u+v)+w = u+(v+w)$ (Associative property of addition)

(d) There is a zero vector 0 in V suchthat for every u in V. we have $(u+0) = u$ (Additive identity)

(e) For every u in V, there is a vector in V denoted by $-u$ suchthat $u+(-u) = 0$ (Additive inverse)

(2) Scalar multiplication.

(a) cu is in V (closure under scalar multiplication)

(b) $c(u+v) = cu + cv$ (Distributive property of scalar multi.)

(c) $(c+d)u = cu + du$ (Distributive property of scalar multi.)

(d) $c(du) = (cd)u$ (Associative property of scalar multi.)

(e) $1(u) = u$ (Scalar identity property)

Eg :- (1) The set $\mathbb{R}$ of real numbers $\mathbb{R}$ is a vector space over $\mathbb{R}$.

(2) The set $\mathbb{R}^2$ of all ordered pairs of real numbers is a vector space over $\mathbb{R}$.

(3) The set $\mathbb{R}^n$ of all ordered n-tuples of real numers is a vector space over $\mathbb{R}$.

(4) The set $M_{m,n}$ of all $m \times n$ matrices, with real entries is a vector space over $\mathbb{R}$.

(5) The set V of all real valued continuous (differentiable or integrable) functions defined on the closed interval $[a,b]$ is a real vector space with the vector addition and scalar multiplication defined as follows.

$$(f+g)(x) = f(x)+g(x)$$

$$(kf)(x) = k f(x), \quad \text{For all } f, g \in V \text{ and } k \in R.$$

<u>Basis</u> :- If V is any vector space and $S = \{v_1, v_2, v_3 \ldots v_n\}$ is a set of vectors in V, then S is called a basis for V if the following two conditions hold.

(i) S is linearly independent

(ii) S spans V.

COMPLEX MATRICES

Conjugate of a Matrix :-

If the elements of matrix A are replaced by their conjugate complexes then the resulting matrix is defined as the conjugate of the given matrix. It is denoted by $\bar{A}$

Eg :- $A = \begin{bmatrix} 7 & 5+4i \\ -2+3i & 4-7i \end{bmatrix}$ $\bar{A} = \begin{bmatrix} 7 & 5-4i \\ -2-3i & 4+7i \end{bmatrix}$

$A = \begin{bmatrix} 0 & 2+3i & 7i \\ 4-7i & 5+3i & 1+i \\ 7 & 1-i & 6+i \end{bmatrix}$ $\bar{A} = \begin{bmatrix} 0 & 2-3i & -7i \\ 4+7i & 5-3i & 1-i \\ 7 & 1+i & 6-i \end{bmatrix}$

Note :- If $\bar{A}$ and $\bar{B}$ be the conjugate matrices of A and B respectively then (i) $\overline{(\bar{A})} = A$

(ii) $\overline{A+B} = \bar{A} + \bar{B}$

(iii) $\overline{KA} = \bar{K}\bar{A}$ where K is complex number.

The transpose of the conjugate of square matrix :-

If A is a square matrix and its conjugate is $\bar{A}$, then the transpose of $\bar{A}$ is $(\bar{A})^T$.

It can be easily seen that $(\bar{A})^T = (\overline{A^T})$ i.e the transpose of the conjugate of a square matrix is same as the conjugate of its transpose.

The transposed conjugate of A is denoted by A^θ.

Eg :- $A = \begin{bmatrix} i & 4+3i \\ 3-i & 7 \end{bmatrix}$ $\bar{A} = \begin{bmatrix} -i & 4-3i \\ 3+i & 7 \end{bmatrix}$ $A^\theta = (\bar{A})^T = \begin{bmatrix} -i & 3+i \\ 4-3i & 7 \end{bmatrix}$

Note :- If A^θ and B^θ be the transposed conjugates of A and B respectively then (i) $(A^\theta)^\theta = A$

(ii) $(A \pm B)^\theta = A^\theta \pm B^\theta$

(iii) $(KA)^\theta = \bar{K} A^\theta$ where k is a complex number

(iv) $(AB)^\theta = B^\theta A^\theta$.

Hermitian Matrix

A square matrix A is said to be hermitian if $A^\theta = A$ i.e $(\bar{A})^T = A$

Eg :- $A = \begin{bmatrix} 5 & 2+4i \\ 2-4i & 7 \end{bmatrix}$ $\bar{A} = \begin{bmatrix} 5 & 2-4i \\ 2+4i & 7 \end{bmatrix}$.

$$A^\theta = (\bar{A})^T = \begin{bmatrix} 5 & 2+4i \\ 2-4i & 7 \end{bmatrix} = A$$

$\therefore$ A is hermitian.

Eg :- $A = \begin{bmatrix} 1 & 1+3i & 2-4i \\ 1-3i & 0 & 5-3i \\ 2+4i & 5+3i & 8 \end{bmatrix}$ $\bar{A} = \begin{bmatrix} 1 & 1-3i & 2+4i \\ 1+3i & 0 & 5+3i \\ 2-4i & 5-3i & 8 \end{bmatrix}$

$$A^\theta = (\bar{A})^T = \begin{bmatrix} 1 & 1+3i & 2-4i \\ 1-3i & 0 & 5-3i \\ 2+4i & 5+3i & 8 \end{bmatrix} = A$$

$\therefore$ A is hermitian.

Note :- (i) The elements of the principal diagonal of a hermitian matrix must be real.

(ii) A Hermitian matrix over the field of real numbers is nothing but a real symmetric matrix.

Skew Hermitian Matrix :-

A square matrix A is said to be skew hermitian if $A^\theta = -A$

Eg :- $A = \begin{bmatrix} 0 & 2-3i \\ -2-3i & i \end{bmatrix}$ $\bar{A} = \begin{bmatrix} 0 & 2+3i \\ -2+3i & -i \end{bmatrix}$

$$A^\theta = (\bar{A})^T = \begin{bmatrix} 0 & -2+3i \\ 2+3i & -i \end{bmatrix}$$

$$A^\theta = -\begin{bmatrix} 0 & 2-3i \\ -2-3i & +i \end{bmatrix}$$

$$A^\theta = -A$$

$\therefore$ A is skew hermitian

Eg :- $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ $\bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$

$$A^\theta = (\bar{A})^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

$$A^\theta = -A$$

$\therefore$ A is skew hermitian.

Note :- (i) The elements of the principal diagonal of a skew hermitian matrix must be purely imaginary or zero.

(ii) A skew hermitian matrix over the field of real numbers is nothing but a real skew symmetric matrix.

Unitary Matrix :-

A square matrix A is said to be unitary if $AA^\theta = A^\theta A = I$.

Eg:- $A = \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$ $\quad \bar{A} = \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$

$$A^\theta = (\bar{A})^T = \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$$

$$AA^\theta = \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix} \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1+3 & \sqrt{3}i - \sqrt{3}i \\ -\sqrt{3}i + \sqrt{3}i & 3+1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^\theta = I$$

$\therefore$ A is unitary matrix.

Note :- A unitary matrix over the field of real numbers is nothing but a real orthogonal matrix.

Properties :-

(i) If A is hermitian then iA is skew hermitian.

(ii) If A is skew hermitian then iA is hermitian.

(iii) The matrix $B^\theta AB$ is hermitian or skew hermitian according as A is hermitian or skew hermitian.

(iv) The transpose of unitary matrix is unitary.

(v) The inverse of unitary matrix is unitary.

(vi) The product of two unitary matrices is unitary.

(vii) The determinant of unitary matrix is of unit modulus.

Properties of Complex matrices :-

Theorem :- If A is a Hermitian then iA is skew Hermitian.

proof :- Let A be a Hermitian matrix so that $A^\theta = A$.

Now $(iA)^\theta = \bar{i} A^\theta$ $\quad [\because (KA)^\theta = \bar{K} A^\theta]$

$= (-i) A^\theta$ $\quad [\because A^\theta = A]$

$= -iA$

$(iA)^\theta = -iA$

$\Rightarrow iA$ is a skew Hermitian matrix.

Eg:- If $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ then prove that A is Hermitian and iA is skew Hermitian.

Sol:- Given $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

$A^\theta = \bar{A}^T = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} = A$

$\therefore A$ is Hermitian.

$iA - i \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

$= \begin{bmatrix} 4i & i+3 \\ i-3 & 7i \end{bmatrix}$

$\overline{iA} = \begin{bmatrix} -4i & -i+3 \\ -i-3 & -7i \end{bmatrix}$

$(\overline{iA})^T = \begin{bmatrix} -4i & -3-i \\ 3-i & -7i \end{bmatrix} = -\begin{bmatrix} 4i & 3+i \\ i-3 & 7i \end{bmatrix}$

$(\overline{iA})^T = (iA)^\theta = -iA$

$\therefore iA$ is skew Hermitian

Theorem :- If A is a skew Hermitian then iA is Hermitian.

Proof :- Let A be a skew Hermitian matrix so that $A^\theta = -A$

$$(iA)^\theta = \bar{i} A^\theta$$

$$= (-i)(-A)$$

$$= iA$$

$$(iA)^\theta = iA$$

iA is Hermitian matrix.

Eg: If $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ then prove that A is skew Hermitian matrix and iA is Hermitian matrix.

Sol:- Given that $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} -3i & -2-i \\ 2-i & i \end{bmatrix}$$

$$A^\theta = \bar{A}^T = -\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} = -A$$

$\therefore A^\theta = -A$

$\therefore$ A is skew Hermitian.

$$iA = i\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} = \begin{bmatrix} -3 & 2i-1 \\ -2i-1 & 1 \end{bmatrix}$$

$$\overline{iA} = \begin{bmatrix} -3 & -2i-1 \\ 2i-1 & 1 \end{bmatrix}$$

$$(\overline{iA})^T = \begin{bmatrix} -3 & 2i-1 \\ -2i-1 & 1 \end{bmatrix}$$

$$(iA)^\theta = (\overline{iA})^T = iA$$

$$(iA)^\theta = iA$$

$\therefore$ iA is ~~skew~~ Hermitian.

Theorem :- The matrix $B^{\theta}AB$ is Hermitian or skew Hermitian according as A is Hermitian or skew Hermitian.

Proof :- (i) Let A be a Hermitian matrix so that $A^{\theta} = A$.

$$\text{Now} \quad (B^{\theta}AB)^{\theta} = B^{\theta}A^{\theta}(B^{\theta})^{\theta}$$
$$= B^{\theta}A^{\theta}B. \quad [\because (B^{\theta})^{\theta} = B]$$
$$= B^{\theta}AB$$
$$(B^{\theta}AB)^{\theta} = B^{\theta}AB.$$
$$\Rightarrow B^{\theta}AB \text{ is a Hermitian matrix.}$$

(ii) Let A be a skew Hermitian matrix so that $A^{\theta} = -A$.

$$(B^{\theta}AB)^{\theta} = B^{\theta}A^{\theta}(B^{\theta})^{\theta}$$
$$= B^{\theta}A^{\theta}B$$
$$= B^{\theta}(-A)B$$
$$= -B^{\theta}AB$$
$$\therefore (B^{\theta}AB)^{\theta} = -B^{\theta}AB.$$
$$\Rightarrow B^{\theta}AB \text{ is skew Hermitian matrix.}$$

Theorem :- The transpose of unitary matrix is unitary.

Proof :- Let A be the unitary matrix so that $AA^{\theta} = I = A^{\theta}A$

$$\text{Now} \quad (AA^{\theta})^{T} = I^{T} = (A^{\theta}A)^{T} \quad (\text{Taking Transpose})$$
$$(A^{\theta})^{T}A^{T} = I = A^{T}(A^{\theta})^{T}$$
$$(A^{T})^{\theta}A^{T} = I$$
$$\Rightarrow A^{T} \text{ is unitary matrix.}$$

Eg:- Prove that $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is unitary matrix and A^{T} is also unitary matrix.

Sol: Given that $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$$\bar{A} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$$

$$A^{\theta} = (\bar{A})^{T} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$AA^\theta = \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is unitary.

$$A^T = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix}$$

$$\overline{A^T} = \frac{1}{2}\begin{bmatrix} 1-i & +1-i \\ -1-i & 1+i \end{bmatrix}$$

$$(A^T)^\theta = (\overline{A^T})^T = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ +1-i & 1+i \end{bmatrix}$$

$$(A^T)^\theta A^T = \frac{1}{4}\begin{bmatrix} 1-i & -1-i \\ +1-i & 1+i \end{bmatrix}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A^T)^\theta A^T = I$$

$\therefore$ A^T is unitary.

Theorem :- The inverse of unitary matrix is unitary.

Proof :- Let A be unitary matrix so that $AA^\theta = I = A^\theta A$

Now $(AA^\theta)^{-1} = I^{-1} = (A^\theta A)^{-1}$ (Taking inverse)

$$(A^\theta)^{-1}(A^{-1}) = I = (A^{-1})(A^\theta)^{-1}$$

$$(A^{-1})^\theta A^{-1} = I = A^{-1}(A^{-1})^\theta$$

$\therefore$ A^{-1} is unitary matrix.

Eg:- If $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ then prove that A and A^{-1} are unitary matrices.

Sol: Given that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad A^\theta = (\bar{A})^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$AA^{\theta} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore\ AA^{\theta} = I$

A is unitary.

$$|A| = \frac{-1}{2} - \frac{1}{2} = -1.$$

$$A^{-1} = \frac{1}{|A|} \operatorname{Adj} A$$

$$A^{-1} = -1 \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\overline{A^{-1}} = - \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(\overline{A^{-1}})^{T} = (A^{-1})^{\theta} = - \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(A^{-1})^{\theta} A^{-1} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore\ (A^{-1})^{\theta} A^{-1} = I.$

$\therefore\ A^{-1}$ is unitary.

Theorem:- The product of two unitary matrices is unitary.

Proof:- Let A and B be two unitary matrices.

$\Rightarrow AA^{\theta} = I = A^{\theta}A$ and $BB^{\theta} = I = B^{\theta}B$

We prove that AB is unitary.

$$\begin{aligned} \text{Consider } (AB)^{\theta}(AB) &= (B^{\theta}A^{\theta})(AB) \\ &= B^{\theta}(A^{\theta}A)B \\ &= B^{\theta}IB \\ &= B^{\theta}B = I \end{aligned}$$

$(AB)^{\theta}(AB) = I$

$\Rightarrow AB$ is unitary

Hence if A and B are unitary then AB is also unitary.

Similarly $(AB)(AB)^{\theta} = (AB)(B^{\theta}A^{\theta})$

$$= A(BB^{\theta})A^{\theta}$$

$$= AIA^{\theta}$$

$$= AA^{\theta} = I.$$

$\therefore (AB)(AB)^{\theta} = (AB)^{\theta}(AB) = I.$

$\therefore$ AB is a unitary matrix.

Eg If $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ and $B = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ are unitary then p/T AB is unitary

Sol: Given that $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ $B = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$AB = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$AB = \frac{1}{\sqrt{6}}\begin{bmatrix} 1+i+1 & +1+i-i \\ -i-1+i & -i+1+1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2+i & 1 \\ -1 & 2-i \end{bmatrix}$$

$$\overline{AB} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2-i & 1 \\ -1 & 2+i \end{bmatrix}$$

$$(AB)^{\theta} = (\overline{AB})^{T} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2-i & -1 \\ 1 & 2+i \end{bmatrix}$$

$$(AB)^{\theta}(AB) = \frac{1}{6}\begin{bmatrix} 2-i & -1 \\ 1 & 2+i \end{bmatrix}\begin{bmatrix} 2+i & 1 \\ -1 & 2-i \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(AB)^{\theta}(AB) = I$$

$\therefore$ AB is unitary

Theorem :- The determinant of a unitary matrix is of unit modulus.

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Proof :- Let A be unitary so that $AA^{\theta} = I$.

$$\Rightarrow |AA^{\theta}| = |I| \qquad \because |AB| = |A||B|$$

$$\Rightarrow |A||A^{\theta}| = 1$$

$$\Rightarrow |A|\,|(\bar{A})^{T}| = 1$$

$$\Rightarrow |A|\,|\bar{A}| = 1 \qquad [\because |B| = |B^{T}|]$$

$$\Rightarrow |A|^{2} = 1$$

$$\Rightarrow |A| \text{ is of unit modulus}.$$

Hence if A is unitary then $|A|$ is of unit modulus.

Eg :- If $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$, then prove that $|A|$ is of unit modulus.

Sol :- Given that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

$$|A| = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

$$|A| = -1$$

$\therefore$ A is unitary and its determinant is of unit modulus.

Eg :- Prove that the determinant of $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is of unit modulus.

Sol : Given that $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$$|A| = \begin{vmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{vmatrix}$$

$$= \frac{2}{4} - \left(-\frac{2}{4}\right)$$

$$|A| = 1$$

$\therefore$ A is unitary and its determinant is of unit modulus.

Theorem :- Every square matrix is uniquely expressed as the sum of Hermitian and skew Hermitian matrices.

Proof :- Let A be a square matrix

Consider $A = \frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A-A^{\theta})$

$A = P + Q$ where $P = \frac{1}{2}(A+A^{\theta})$ $Q = \frac{1}{2}(A-A^{\theta})$.

We prove that P is Hermitian and Q is skew Hermitian matrices.

$$P = \frac{1}{2}(A+A^{\theta})$$

$$P^{\theta} = \left[\frac{1}{2}(A+A^{\theta})\right]^{\theta} = \frac{1}{2}(A+A^{\theta})^{\theta}$$

$$= \frac{1}{2}\left[A^{\theta} + (A^{\theta})^{\theta}\right]$$

$$= \frac{1}{2}(A^{\theta}+A)$$

$$P^{\theta} = P.$$

$\therefore$ P is Hermitian matrix.

$$Q = \frac{1}{2}(A-A^{\theta})$$

$$Q^{\theta} = \left[\frac{1}{2}(A-A^{\theta})\right]^{\theta}$$

$$= \frac{1}{2}(A-A^{\theta})^{\theta}$$

$$= \frac{1}{2}(A^{\theta}-(A^{\theta})^{\theta})$$

$$= \frac{1}{2}(A^{\theta}-A)$$

$$= -\frac{1}{2}(A-A^{\theta})$$

$$Q^{\theta} = -Q$$

$\therefore$ Q is skew Hermitian matrix

Thus every square matrix can be expressed as the sum of Hermitian and skew Hermitian matrices.

Uniqueness :-

Let $A = R+S$ be another such representation of A, where R is Hermitian and Q is skew Hermitian.

Then we have to P.T $P=R$ and $Q=S$.

$$P=\frac{1}{2}(A+A^{\theta})$$

$$=\frac{1}{2}\left[(R+S)+(R+S)^{\theta}\right]$$

$$=\frac{1}{2}\left[(R+S)+(R^{\theta}+S^{\theta})\right]$$

$$=\frac{1}{2}\left[R+S+R-S\right]=\frac{1}{2}(2R)$$

$$P=R.$$

$$Q=\frac{1}{2}(A-A^{\theta})$$

$$=\frac{1}{2}\left[(R+S)-(R+S)^{\theta}\right]$$

$$=\frac{1}{2}\left[(R+S)-(R^{\theta}+S^{\theta})\right]$$

$$=\frac{1}{2}\left[(R+S)-(R-S)\right]$$

$$=\frac{1}{2}(2S)$$

$$Q=S$$

$\therefore$ $P=R$ and $Q=S$

Hence the representation is unique.

(1) Express the matrix $A=\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of a Hermitian and a skew Hermitian matrices.

Sol:- Given that $A=\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$

$$\bar{A}=\begin{bmatrix} -i & 2+3i & 4-5i \\ 6-i & 0 & 4+5i \\ i & 2+i & 2-i \end{bmatrix}$$

$$A^{\theta}=(\bar{A})^{T}=\begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

Hermitian part of the matrix A is $P=\frac{1}{2}(A+A^{\theta})$

$$P=\frac{1}{2}(A+A^{\theta})=\frac{1}{2}\left\{\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}+\begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}\right\}$$

$$P=\frac{1}{2}\begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}$$

This is a Hermitian matrix

$$Q=\frac{1}{2}(A-A^{\theta})=\frac{1}{2}\left\{\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}-\begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}\right\}$$

$$Q=\frac{1}{2}\begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

This is a skew Hermitian matrix.

$$P+Q=\frac{1}{2}\begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} 2i & 4-6i & 8+10i \\ 12+2i & 0 & 8-10i \\ -2i & 4-2i & 4+2i \end{bmatrix}$$

$$=\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}=A$$

$\therefore$ $A=P+Q$ Where P is Hermitian and Q is skew Hermitian.

Prove that every Hermitian matrix can be written as $P+iQ$ where P is a real symmetric matrix and Q is a real skew symmetric matrix. 2.9

Sol:- Let A be a Hermitian matrix.

$$A^{\theta} = A$$

$$A = \frac{1}{2}(A+\bar{A}) + i\,\frac{1}{2i}(A-\bar{A}) = P+iQ.$$

Where $P = \frac{1}{2}(A+\bar{A})$ and $Q = \frac{1}{2i}(A-\bar{A})$ are real matrices.

$$P^{T} = \left[\frac{1}{2}(A+\bar{A})\right]^{T} = \frac{1}{2}\left[A^{\theta}+\bar{A}\right]^{T}$$

$$= \frac{1}{2}\left[(\bar{A})^{T}+\bar{A}\right]^{T} = \frac{1}{2}\left[\left[(\bar{A})^{T}\right]^{T}+(\bar{A})^{T}\right]$$

$$= \frac{1}{2}\left(\bar{A}+A^{\theta}\right) = \frac{1}{2}(\bar{A}+A)$$

$$P^{T} = P.$$

Hence P is a real symmetric matrix.

Also, $$Q^{T} = \left[\frac{1}{2i}(A-\bar{A})\right]^{T} = \frac{1}{2i}\left[A^{\theta}-\bar{A}\right]^{T}$$

$$= \frac{1}{2i}\left[(\bar{A})^{T}-\bar{A}\right]^{T} = \frac{1}{2i}\left[\left[(\bar{A})^{T}\right]^{T}-(\bar{A})^{T}\right]$$

$$= \frac{1}{2i}\left[\bar{A}-A^{\theta}\right] = \frac{1}{2i}(\bar{A}-A)$$

$$= -\frac{1}{2i}(A-\bar{A}) = -Q$$

$$Q^{T} = -Q.$$

Hence Q is a real skew symmetric matrix.

Thus, every Hermitian matrix can be written as $P+iQ$, where P is a real symmetric matrix and Q is a real skew symmetric matrix.

Express the Hermitian matrix $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$ as $P + iQ$ where P is a real symmetric matrix and Q is a real skew symmetric matrix.

sol: Given that $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}$$

$$A = \frac{1}{2}(A+\bar{A}) + i\,\frac{1}{2i}(A-\bar{A}) = P + iQ.$$

where $P = \frac{1}{2}(A+\bar{A})$, $Q = \frac{1}{2i}(A-\bar{A})$

Let $P = \frac{1}{2}(A+\bar{A}) = \frac{1}{2}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} + \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\}$

$$P = \frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$Q = \frac{1}{2i}(A-\bar{A}) = \frac{1}{2i}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2+3i \\ 1-i & 2+3i & 2 \end{bmatrix} - \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\}$$

$$Q = \frac{1}{2i}\begin{bmatrix} 0 & -2i & 2i \\ 2i & 0 & -6i \\ -2i & 6i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$P^T = P, \quad Q^T = -Q$$

We know that P is a real symmetric matrix and Q is a real skew symmetric matrix.

$$A = P + iQ = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} + i\begin{bmatrix} 0 & -i & i \\ i & 0 & -3i \\ -i & 3i & 0 \end{bmatrix}$$

Show that every square matrix can be uniquely expressed as $P+iQ$ where P and Q are Hermitian matrices. 26

Sol: Let A be a square matrix.

$$A = \frac{1}{2}(A+A^\theta) + i\frac{1}{2i}(A-A^\theta) = P+iQ.$$

where $P = \frac{1}{2}(A+A^\theta)$ and $Q = \frac{1}{2i}(A-A^\theta)$

Now $$P^\theta = \frac{1}{2}(A+A^\theta)^\theta = \frac{1}{2}\left[A^\theta + (A^\theta)^\theta\right]$$
$$= \frac{1}{2}(A^\theta + A) = P$$

$$P^\theta = P$$

Hence, P is a Hermitian matrix.

$$Q^\theta = \left[\frac{1}{2i}(A-A^\theta)\right]^\theta = -\frac{1}{2i}\left[A^\theta - (A^\theta)^\theta\right]$$
$$= -\frac{1}{2i}\left[A^\theta - A\right]$$
$$= \frac{1}{2i}\left[A - A^\theta\right]$$

$$Q^\theta = Q.$$

Hence Q is a Hermitian matrix.

Thus, every square matrix can be expressed as $P+iQ$ where P and Q are Hermitian matrices

<u>Uniqueness</u>:- Let $A = R+iS$ where R and S are Hermitian matrices.

$$A^\theta = (R+iS)^\theta = R^\theta + (iS)^\theta = R - iS \qquad R^\theta = R, \quad S^\theta = S.$$

$$\frac{1}{2}(A+A^\theta) = \frac{1}{2}\left[(R+iS)+(R-iS)\right] = R = P$$

$$\frac{1}{2}(A-A^\theta) = \frac{1}{2}\left[(R+iS)-(R-iS)\right] = iS = iQ.$$

Hence, representation $A = P+iQ$ is unique.

Express the matrix $A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$ as $P + iQ$ where P and Q are both Hermitian.

sol:

$$A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -2i & -3 & 1+i \\ 0 & 2-3i & 1-i \\ 3i & 3-2i & 2+5i \end{bmatrix} \qquad A^{\theta} = \bar{A}^{T} = \begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}$$

Let $P = \frac{1}{2}(A + A^{\theta}) = \frac{1}{2}\left\{\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix} + \begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 0 & -3 & 1+2i \\ -3 & 4 & 4-i \\ 1-2i & 4+i & 4 \end{bmatrix}$$

$Q = \frac{1}{2i}(A - A^{\theta}) = \frac{1}{2i}\left\{\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix} - \begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}\right\}$

$$= \frac{1}{2i}\begin{bmatrix} 4i & -3 & 1-4i \\ 3 & 6i & -2+3i \\ -1-4i & 2+3i & -10i \end{bmatrix}$$

We know that P and Q are Hermitian matrices.

$$A = P + iQ = \frac{1}{2}\begin{bmatrix} 0 & -3 & 1+2i \\ -3 & 4 & 4-i \\ 1-2i & 4+i & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4i & -3 & 1-4i \\ 3 & 6i & -2+3i \\ -1-4i & 2+3i & -10i \end{bmatrix}$$

Prove that every skew Hermitian matrix can be written as $P+iQ$ where P is a real skew symmetric matrix and Q is a real symmetric matrix.

Sol: Let A be a skew Hermitian matrix.

$$A^{\theta} = -A$$

$$A = \frac{1}{2}(A+\bar{A}) + i\,\frac{1}{2i}(A-\bar{A}) = P+iQ$$

Where $P = \frac{1}{2}(A+\bar{A})$ and $Q = \frac{1}{2i}(A-\bar{A})$ are real matrices.

$$P^T = \left[\frac{1}{2}(A+\bar{A})\right]^T = \frac{1}{2}\left[-A^{\theta}+\bar{A}\right]^T$$

$$= \frac{1}{2}\left[-(\bar{A})^T + \bar{A}\right]^T$$

$$= \frac{1}{2}\left[-[(\bar{A})^T]^T + (\bar{A})^T\right]$$

$$= \frac{1}{2}\left[-\bar{A} + A^{\theta}\right]$$

$$= \frac{1}{2}\left[-\bar{A} - A\right] = -\frac{1}{2}(A+\bar{A}) = -P$$

$$P^T = -P$$

Hence P is a real skew symmetric matrix.

$$Q^T = \left[\frac{1}{2i}(A-\bar{A})\right]^T = \frac{1}{2i}\left(-A^{\theta}-\bar{A}\right)^T$$

$$= \frac{1}{2i}\left[-(\bar{A})^T - \bar{A}\right]^T = \frac{1}{2i}\left[-[(\bar{A})^T]^T - (\bar{A})^T\right]$$

$$= \frac{1}{2i}\left[-\bar{A} - A^{\theta}\right] = \frac{1}{2i}\left[-\bar{A}+A\right] = \frac{1}{2i}(A-\bar{A}) = Q$$

$$Q^T = Q.$$

Hence Q is a real symmetric matrix.

Thus, every skew Hermitian matrix can be written as $P+iQ$ where P is a real skew symmetric matrix and Q is a real symmetric matrix.

Express the skew Hermitian matrix $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ as $P+iQ$ where

P is a real skew symmetric matrix and Q is a real symmetric matrix.

Sol:

$$A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}$$

Let $P = \frac{1}{2}(A+\bar{A}) = \frac{1}{2}\left\{\begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1+i & 3i & 0 \end{bmatrix} + \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}\right\}$

$$P = \frac{1}{2}\begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$Q = \frac{1}{2i}(A-\bar{A}) = \frac{1}{2i}\left\{\begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1+i & 3i & 0 \end{bmatrix} - \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{2i}\begin{bmatrix} 4i & 2i & -2i \\ 2i & -2i & 6i \\ -2i & 6i & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$P^T = -P, \quad Q^T = Q.$$

We know that P is a real skew symmetric matrix and Q is a real symmetric matrix.

$$A = P + iQ = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2i & i & -i \\ i & -i & 3i \\ -i & 3i & 0 \end{bmatrix}.$$

COMPLEX MATRICES

1 Define complex matrix. Give an example.

2 Define Conjugate of a matrix. Give an example.

3 Define Conjugate transpose of a matrix Give an example.

4 Define Hermitian matrix Givea an example.

5 Define skew hermitial matrix Give an example.

6 Define Unitary matrix. Give an example.

7 (a) If A is Hermitiah matrix then prove that iA is skew Hermitian. matrix

(b) If $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & 5i & 2 \end{bmatrix}$ show that A is a Hermitian matrix and B=iA is a skew Hermitian matrix.

8 (a) If A is skew Hermitian matrix then prove that iA is Hermitian matrix.

(b) If $A = \begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$ show that A is skew Hermitian matrix and B=iA is a Hermitian matrix.

9. Express the matrix $A = \begin{bmatrix} 1+i & -i & 2-3i \\ 2 & 1+2i & 3+i \\ -1+i & 3 & 1-2i \end{bmatrix}$ as the sum of a Hermitian matrix and a skew Hermitian matrix

Ans:- $P = \frac{1}{2}\begin{bmatrix} 2 & 2-i & 1-4i \\ 2+i & 2 & 6+i \\ 1+4i & 6-i & 2 \end{bmatrix}$ $Q = \frac{1}{2}\begin{bmatrix} 2i & -i-2 & 3-2i \\ 2-i & 4i & i \\ -3-2i & i & -4i \end{bmatrix}$

10 (a) show that the matrix $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is both a skew Hermitian matrix and a unitary matrix.

(b) verify that the matrix $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

11 (a) show that the matrix $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if $a^2+b^2+c^2+d^2=1$.

(b) If $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$ show that AA^* is a Hermitian matrix.

12 If $A=\begin{bmatrix}0 & 1+2i\\ -1+2i & 0\end{bmatrix}$ show that $B=(I-A)(I+A)^{-1}$ is a unitary matrix !!

13 Find the Eigen values and Eigen vectors of the matrix $A=\begin{bmatrix}4 & 1-3i\\ 1+3i & 7\end{bmatrix}$

Ans :- $\lambda = 9, 2$ $\quad X_1 = \begin{bmatrix}-1+3i\\ 2\end{bmatrix}$ $\quad X_2 = \begin{bmatrix}1-3i\\ 5\end{bmatrix}$

14 Find the Eigen values and Eigen vectors of the matrix $A=\begin{bmatrix}2i & 3i\\ 3i & 0\end{bmatrix}$

Ans:- $\lambda = 1+\sqrt{10}\,i,\ 1-\sqrt{10}\,i$

15 Find the Eigen values and Eigen vectors of the matrix $A=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$

Ans :- $\lambda = 1, -1$ $\quad x_1 = \begin{bmatrix}1\\ i-i\sqrt{2}\end{bmatrix}$, $\quad x_2 = \begin{bmatrix}1\\ i+i\sqrt{2}\end{bmatrix}$

MODULE -II

EIGEN VALUES AND EIGEN VECTORS

EIGEN VALUES AND EIGEN VECTORS.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$ be a square matrix. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ be a.

column vector. Consider the equation $AX = \lambda X$ —— ①. Where λ is a. Scalar. If I is a unit matrix of order n then the equation ① can be written as $AX = \lambda I X$.

$$AX - \lambda IX = 0.$$

$$(A - \lambda I)X = 0 \quad \text{—— ②}.$$

This matrix equation represents the following system of n homogeneous equations in n unknowns.

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0. \\ a_{31}x_1 + a_{32}x_2 + (a_{33}-\lambda)x_3 + \cdots + a_{3n}x_n &= 0. \\ - - - - - - - - - - - - & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right] \text{—— ③}$$

Here the Co efficient matrix of this system is $A - \lambda I$.

We know that The necessary and sufficient condition for the system ③ posseses a non zero solution is that the coefficient matrix $A - \lambda I$ is singular i.e $|A - \lambda I| = 0$.

<u>Characteristic Matrix</u> :- Let A be a square matrix of order n and I be a unit matrix of order n Then the matrix $A - \lambda I$ is called characteristic matrix where λ is a constant.

Characteristic Polynomial :—

The determinant of the matrix $A-\lambda I$ is called characteristic polynomial in λ of degree n.

Characteristic Equation :—

For a square matrix A, the equation $|A-\lambda I|=0$ is called the characteristic equation.

Eigen values :- The roots of the characteristic equation are called the characteristic values or roots or Eigen values or Latent roots or proper values of the square matrix.

Note :- The set of the Eigen values of A is called the spectrum of A.

Eigen vectors :- If λ is an eigen value of the square matrix A then $\det(A-\lambda I)=0$ i.e The matrix $A-\lambda I$ is singular. Therefore there exists a non zero vector x such that $(A-\lambda I)x=0$ or $Ax=\lambda x$ is said to be the eigen vector or characteristic vector of A corresponding to the eigen ~~values~~. values.

(OR)

Let A be a square matrix of order n. A Non Zero vector x is said to be characteristic vector of A if there exists a scalar λ such that $Ax=\lambda x$.

Note :- An Eigen value of a square matrix A can be zero. But a zero vector can not be an Eigen vector of A.

Properties of Eigen values and Eigen vectors :-

1) The sum of the Eigen values of a square matrix is equal to its trace of the matrix.

i.e If $\lambda_1, \lambda_2, \lambda_3$ are Eigen values of A then $tr(A) = \lambda_1 + \lambda_2 + \lambda_3$.

Eg:-(i) If 2, 3, 5 are Eigen values of A then $tr(A) = 2+3+5 = 10$

(ii) If 0, 1, -1 are Eigen values of A then $tr(A) = 0+1-1 = 0$.

2) The product of the Eigen values of a square matrix is equal to its determinant.

i.e If $\lambda_1, \lambda_2\, \lambda_3$ are Eigen values of A then $|A| = \lambda_1 \lambda_2 \lambda_3$.

Eg:- (i) If 0, 0, 1 are Eigen values of A then $|A| = 0.0.1 = 0$.

(ii) If 1, 3, -5 are Eigen values of A then $|A| = 1.3.(-5) = -15$.

Note :-(i) If one of the Eigen values of A is zero then A is singu-lar matrix

(ii) If all the Eigen values of A are non zero then A is non singu-lar matrix.

3) If λ is an eigen value of A corresponding to the eigen vector X. then λ^n is an eigen value of A^n corresponding to the eigen vector X.

Eg :- If -1, 1, 2 are Eigen values of A then Eigen values of A^3 are $(-1)^3$, 1^3 and 2^3 i.e -1, 1, 8.

4) If λ is an eigen value of A corresponding to the eigen vector X. then $K\lambda$ is an eigen value of KA corresponding to the eigen vector X. Where k is non zero scalar.

Eg:- If 1, 2, 3 are Eigen values of A then Eigen values of 3A are 3, 6, and 9.

5) If λ is an eigen value of a non singular matrix A correspo-
-nding to the eigen vecto X, then λ^{-1} is an eigen value of A^{-1}.
corresponding to the eigen vector X.

Eg:- If 1, 2, 3 are eigen values of A then eigen values of
A^{-1} are 1^{-1}, 2^{-1} and 3^{-1} i.e 1, $\frac{1}{2}$ and $\frac{1}{3}$.

6) If λ is an eigen value of a non singular matrix A correspon-
-ding to the eigen vector X then $\frac{|A|}{\lambda}$ is an eigen value of the
matrix adjA corresponding to the eigen vector X.

Eg:- If 1, 3, 5 are eigen values of A then eigen values of
adj A are given by $\frac{|A|}{\lambda} = \frac{15}{1}, \frac{15}{3}, \frac{15}{3}$ i.e 15, 3, 5
$[\because |A| = 1.3.5 = 15]$.

7) If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also
an eigen value.

8) The Eigen values of a triangular matrix are just the diagonal
element of the matrix.

Eg:- An Eigen values of $A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$ are $\lambda = 1, -3$.

9) For a real symmetric matrix, the eigen vectors corresponding
to two distinct eigen values are orthogonal.

Eg:- $X_1 = [-16 \ 1 \ 11]^T$ $X_2 = [2 \ -1 \ 3]^T$ $X_3 = [1 \ 5 \ 1]^T$ are eigen.
vectors of corresponding to distinct eigen values of real
symmetric matrix. Here X_1, X_2 and X_3 are pairwise orthogonal.

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10) If X is an Eigen vector of a matrix A, then X can not correspond to more than one eigen value of A.

11) The Eigen vectors corresponding to distinct eigen values of a matrix are linearly independent.

12) If X_1 and X_2 are two Eigen vectors of a matrix A corresponding to some some eigen value λ then any linear combination $K_1X_1 + K_2X_2$ where K_1, K_2 are arbitrary constants is also an eigen vector of A corresponding to the same Eigen value λ.

13) A square matrix A and its transpose A^T have the same eigen values.

Eg:- If 2, 3 are eigen values of A then eigen values A^T are 2,3.

14) If λ is an eigen value of the matrix A then $\lambda + K$ is an eigen value of the matrix $A + KI$ corresponding to the eigen vector X.

Eg: (i) If 1, 2, 3 are eigen values of A then eigen values of $A + 2I$ are $1+2, 2+2, 3+2$ i.e 3, 4 and 5.

(ii) If 0, 1, -2 are eigen values of A then eigen values of $A - 3I$ are $0-3, 1-3, -2-3$ i.e $-3, -2$ and -5.

15) An Eigen values of skew hermitian matrix are purely imaginary or zero.

16) An Eigen values of hermitian matrix are real.

17) The Eigen values of an unitary matrix have absolute value 1.

Working procedure to find Eigen values :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

step(i) :- The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$.

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \text{ ——— } (1)$$

Where S_1 = Sum of the principal diagonal elements of A i.e $tr(A)$

$S_1 = a_{11} + a_{22} + a_{33}$.

S_2 = Sum of the minors of principal diagonal elements of A

$S_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$S_3 = \det A$.

step(ii) :- Find S_1, S_2 and S_3,

step(iii) :- substitute the values of S_1, S_2 and S_3 in (1)

solve the eqn. (1), we get Eigen values λ_1, λ_2 and λ_3.

If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(a) verify that $|A| = \lambda_1 \lambda_2 \lambda_3$ and $tr(A) = \lambda_1 + \lambda_2 + \lambda_3$.

(b) Find the eigen values for the following matrices

(i) A (ii) A^T (iii) A^{-1} (iv) $4A^{-1}$ (v) A^2 (vi) $A^2 - 2A + I$ (vii) $A^3 + 2I$.

(viii) $A - 2I$.

Sol:- Given that $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$.

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0.$$

Where S_1 = sum of the principal diagonal elements of $A = 3+5+3 = 11$

S_2 = sum of the minors of principal diagonal elements of A

$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$

$= (15-1) + (9-1) (15-1)$

$S_2 = 36$

$S_3 = |A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3(15-1) + 1(-3+1) + 1(1-5)$

$S_3 = 36$.

$\therefore$ The characteristic equation of A is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\lambda = 2, 3, 6$.

(a) $|A| = 2.3.6 = 36$, $tr(A) = 2+3+6 = 11$.

(b) (i) Eigen values of $A = \lambda$ $\longrightarrow$ $2, 3, 6$

(ii) Eigen values of $A^T = \lambda$ $\longrightarrow$ $2, 3, 6$

(iii) Eigen values of $A^{-1} = \lambda^{-1}$ $\longrightarrow$ $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

(iv) Eigen values of $4A^{-1} = 4\lambda^{-1}$ $\longrightarrow$ $\frac{4}{2}, \frac{4}{3}, \frac{4}{6}$

(v) Eigen values of $A^2 = \lambda^2$ $\longrightarrow$ $2^2, 3^2, 6^2$

(vi) Eigen values of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$ $\longrightarrow$ $1, 4, 25$

(vii) Eigen values of $A^3 + 2I = \lambda^3 + 2$ $\longrightarrow$ $10, 29, 218$

(viii) Eigen values of $A - 2I = \lambda - 2$ $\longrightarrow$ $0, 1, 4.$

Working procedure to find Eigen values and Eigen vectors :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Step(i) :- The characteristic equation of A is $|A-\lambda I| = 0$.

i.e $$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0.$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad \text{——} \quad (1)$$

Where $S_1 = tr(A)$

S_2 = sum of the minors of principal diagonal elements of A

$S_3 = |A|$.

Step (ii):- Solve the characteristic eqn. (1), we get Eigen values λ_1, λ_2 and λ_3.

Step (iii) For finding an Eigen vector corresponding Eigen value $\lambda = \lambda_1$ we solve homogeneous system $(A-\lambda I)X = 0$. —— (2).

i.e $$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Similarly we can find an eigen vector corresponding eigen value $\lambda = \lambda_2$, $\lambda = \lambda_3$. by solving homogeneous system (2).

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→ Determine the characteristic roots and the characteristic vectors of the matrix $A=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$. Also find characteristic roots and char. vectors of (i) A^2 (ii) A^{-1}.

Sol:- Given that $A=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$

Let λ be the eigen value of A.

The characteristic equation of A is $|A-\lambda I|=0$.

$$\text{i.e} \begin{vmatrix}2-\lambda&1&0\\0&2-\lambda&1\\0&0&2-\lambda\end{vmatrix}=0$$

$$(2-\lambda)^3=0$$

$$\lambda=2,2,2.$$ [Algebraic multiplicity of $\lambda=2$ is 3]

∴ Eigen values of A are $\lambda=2,2,2$.

Now the Eigen vector $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ of A corresponding to the Eigen value λ are obtained by solving the homogeneous system of eqns

$(A-\lambda I)X=0$ i.e $\begin{bmatrix}2-\lambda&1&0\\0&2-\lambda&1\\0&0&2-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ —— ①

Eigen vector corresponding to Eigen value $\lambda=2$:-

For $\lambda=2$, The system ① can be written as

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Which is in echelon form.

Here rank of the coefficient matrix of the system is 2 i.e $r=2$

Sothat the system has $n-r=3-2=1$ L.I solution.

There is only one L.I eigen vector corresponding to Eigen value $\lambda=2$.

To determine this, We have to assign an arbitrary value for $n-r = 3-2 = 1$ variable.

From the above system, the eqn's can be written as.

$$x_2 = 0 \ , \ x_3 = 0 \ .$$

Note that we can not find x_1 from these eqn's. As x_1 is not present in any of these equations, it follows that x_1 can be arbitrary

Hence $x_1 = K_1 \ x_2 = 0, \ x_3 = 0$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K_1 \\ 0 \\ 0 \end{bmatrix}$ is the only linearly independent Eigen vector of A Corresponding to the Eigen value $\lambda = 2$. (Geometric multiplicity of $\lambda = 2$ is 1).

(i) We know that If λ is an eigen value of A corresponding to the Eigen vector X then λ^n is an eigen value of A^n Corresponding to the Eigen vector X.

$\therefore$ Eigen values of A^2 is $\lambda^2 = 2^2, 2^2, 2^2$ and the corresponding eigen vector is $X_1 = \begin{bmatrix} K_1 \\ 0 \\ 0 \end{bmatrix}$.

(ii) We know that If λ is an eigen value of A corresponding to the Eigen vector X then λ^{-1} is an eigen value of A^{-1} corresponding to the Eigen vector X.

$\therefore$ Eigen values of A^{-1} is $\lambda^{-1} = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ and the corresponding Eigen vector is $X_1 = \begin{bmatrix} K_1 \\ 0 \\ 0 \end{bmatrix}$.

→ Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$.
Also find eigen values and eigen vectors of
(i) adj A (ii) $A - 3I$.

Sol:- Given that $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(-2-\lambda) = 0$$

$$\lambda = 1, 2, -2$$

∴ Eigen values of the matrix A are $\lambda = 1, 2, -2$

[Algebraic multiplicity of $\lambda = 1, 2, -2$ is 1]

Now the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to Eigen value λ are obtained by solving the homogeneous system $(A - \lambda I)X = 0$.

$$\text{i.e} \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—} (1)$$

(i) <u>Eigen vector corresponding to the Eigen value $\lambda = 1$</u> :-

For $\lambda = 1$, The system (1) can be written as

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by applying E-row operations only. and hence determine the rank of coeff. matrix.

$$R_2 \to 2R_2 - R_1$$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \rightarrow 5R_3 + 3R_2$$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form

Here the rank of the coeff. matrix of the system is 2 i.e $r=2$

Sothat the system has $n-r = 3-2 = 1$ L.I solution.

There is only one L.I eigen vector corresponding to eigen value $\lambda=1$.

To determine this we have to assign an arbitrary value for $n-r=$ $3-2=1$ variable.

From the above system, the eqn's can be written as

$$2x_2 - x_3 = 0$$

$$5x_3 = 0 \Rightarrow x_3 = 0$$

$$x_2 = \frac{x_3}{2}$$

$$x_2 = 0$$

Now we can't find x_1 from these equations. As x_1 is not present in any of these eqn's. it follows that x_1 is an arbitrary.

Hence $x_1 = k_1,\ x_2 = 0,\ x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ where } k_1 \neq 0.$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the linearly independent eigen vector corresponding to eigen value $\lambda = 1$.

[Geometric multiplicity of $\lambda=1$ is 1]

Case (ii) Eigen vector corresponding to Eigen value $\lambda = 2$:-

For $\lambda = 2$, The system (1) can be written as

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the co eff. matrix into echelon form by applying E-row operations only and hence determine the rank of coeff. matrix

$$R_3 \to R_3 + 2R_2$$

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the Coeff. matrix of the system is 2 i.e $r = 2$

Sothat the system has $n - r = 3 - 2 = 1$ L.I solution.

There is only one L.I eigen vector Corresponding to the eigen value $\lambda = 2$.

To determine this we have to assign an arbitrary value for $n - r = 3 - 2 = 1$ variable.

From the above system, the eqn's can be written as

$$-x_1 + 2x_2 - x_3 = 0$$

$$2x_3 = 0 \Rightarrow x_3 = 0$$

$$-x_1 + 2x_2 = 0$$

$$\Rightarrow x_1 = 2x_2$$

choose $x_2 = k_2$ Then $x_1 = 2k_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_2 \\ k_2 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

$\therefore X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is the L.I eigen vector corresponding eigen value $\lambda = 2$.

[Geometric multiplicity of $\lambda = 2$ is 1]

Case (iii) Eigen vector corresponding to the Eigen value $\lambda = -2$:-

For $\lambda = -2$, The system ① can be written as

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the coeff. matrix of the system is 2 i.e $r = 2$

So that the system has $n - r = 3 - 2 = 1$ L.I solution.

There is only one L.I eigen vector corresponding to eigen value $\lambda = -2$.

To determine we have to assign an arbitrary value for $n - r =$ $3 - 2 = 1$ variable.

From the above system, the equations can be written as

$$3x_1 + 2x_2 - x_3 = 0$$

$$4x_2 + 2x_3 = 0 \implies x_2 = -\frac{1}{2} x_3$$

choose $x_3 = k_3$

$$x_2 = -\frac{1}{2} k_3$$

$$x_1 = \frac{x_3 - 2x_2}{3}$$

$$x_1 = \frac{2}{3} k_3.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}k_3 \\ -\frac{1}{2}k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \text{ where } k_3 \neq 0.$$

$\therefore X_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = -2$

[Geometric multiplicity of $\lambda = -2$ is 1]

$\therefore$ The Eigen values of A are $1, 2, -2$ and the corresponding eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2/3 \\ -1/2 \\ 1 \end{bmatrix}$.

(i) We know that λ is an eigen value of non singular matrix A. Corresponding to the eigen vector x then $\frac{|A|}{\lambda}$ is an eigen value of adj A corresponding to the eigen vector X.

$\therefore$ Eigen values of adj A are $\frac{|A|}{\lambda} = \frac{-4}{1}, \frac{4}{2}, \frac{4}{-2}$ i.e $-4, 2, -2$.

and corresponding eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} \frac{2}{3} \\ -1/2 \\ 1 \end{bmatrix}$

(ii) We know that λ is an eigen value of A corresponding to the eigen vector x then $\lambda - k$ is an eigen value of $A - kI$ corresponding to the eigen vector X.

$\therefore$ Eigen values of $A - 3I$ are $\lambda - 3 = -2, -1, -5$ and corresponding eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2/3 \\ -1/2 \\ 1 \end{bmatrix}$.

Find the eigen values and the corrosponding eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Sol:- Given that $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$.

i.e $$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2$

$$\begin{vmatrix} -\lambda & \lambda & 0 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$R_3 \to R_3 - R_2$

$$\begin{vmatrix} -\lambda & \lambda & 0 \\ 1 & 1-\lambda & 1 \\ 0 & \lambda & -\lambda \end{vmatrix} = 0.$$

$$\lambda^2 \begin{vmatrix} -1 & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -1 \end{vmatrix} = 0$$

$C_2 \to C_2 + C_3$

$$\lambda^2 \begin{vmatrix} -1 & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & -1 \end{vmatrix} = 0.$$

$$\lambda^2 \left[[-1(\lambda-2) - 0] - (-1) \right] = 0.$$

$$\lambda^2 [2 - \lambda + 1] = 0 \Rightarrow \lambda^2 (3 - \lambda) = 0$$

$$\lambda = 0, 0, 3.$$

The Eigen values of the matrix A are $\lambda = 0, 0, 3$.

Now the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to Eigen value λ

are obtained by solving the homogeneous system $(A-\lambda I)X = 0$.

$$\text{i.e} \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{———} (1)$$

Eigen vector corresponding to the Eigen value $\lambda = 0$:—

For $\lambda = 0$, The system (1) can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and hence determine the rank of the coefficient matrix.

$$R_2 \longrightarrow R_2 - R_1, \quad R_3 \longrightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the Rank of the coefficient matrix of the system is 1 i.e $r = 1$.

So that the system has $n - r = 3 - 1 = 2$ linearly independent sol's.

There are two linearly independent eigen vectors corresponding to the eigen value $\lambda = 0$.

To determine this, from the above system. the eqns can be written as $x_1 + x_2 + x_3 = 0$

Choose $x_2 = k_1$
$x_3 = k_2$

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$$x_1 = -x_2 - x_3$$

$$x_1 = -k_1 - k_2.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are the L.I eigen vectors corrosponding to the eigen value $\lambda = 0$.

Eigen vector corrosponding to the eigen value $\lambda = 3$:—

For $\lambda = 3$, The system (1) can be written as.

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and. hence determine the rank of the coefficient matrix.

$$R_2 \to 2R_2 + R_1,\ R_3 \to 2R_3 + R_1$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the Rank of the coefficient matrix of the system is 2 i.e $r = 2$

so that the system has $n - r = 3 - 2 = 1$ L.I sol.

There is only one L.I eigen vector corrosponding to the eigen value $\lambda = 3$.

To determine this, from the above system, the eqn's can be written as $-2x_1 + x_2 + x_3 = 0$

$$x_2 - x_3 = 0$$

choose $x_3 = k_1$

$$x_2 = x_3$$

$$x_2 = k_1$$

$$2x_1 = x_2 + x_3$$

$$x_1 = k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{where } k_1 \neq 0$$

$x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the L.I eigen vector corresponding to the eigen value. $\lambda = 3$.

$\therefore$ The Eigen value of A are 0, 0, 3 and the corresponding to the eigen vectors are $x_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(2) show that $A=\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is a skew Hermitian matrix and also

Find eigen values and the corresponding eigen vectors of A.

sol:- Given that $A=\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

$$\bar{A}=\begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$A^{\theta}=\bar{A}^{T}=\begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$A^{\theta}=-\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}-A$$

$\therefore$ A is skew hermitian matrix.

The characteristic equation of A is $|A-\lambda I|=0$

$$\text{i.e} \begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{vmatrix}=0$$

$$(i-\lambda)(\lambda^2+1)=0$$

$$\lambda=-i, i, i$$

The eigen values of the matrix A are $\lambda=-i, i, i$

Now we have to find eigen vectors $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corresponding to the eigen values of λ by solving the homogeneous system $(A-\lambda I)X=0$.

$$\text{i.e} \begin{bmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{---} (1).$$

Case(ii):- Eigen vector corresponding to the eigen value $\lambda = -i$

For $\lambda = -i$, The system ① can be written as

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and determine the rank of the matrix.

$R_3 \longrightarrow R_3 - R_2$

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the rank of the coefficient matrix of the system is $r = 2$ = The No. of non zero rows.

So that the system have $n - r = 3 - 2 = 1$ linearly independent sol.

$\therefore$ There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = -i$

To determine this, we have to assign an arbitrary value for $n - r = 3 - 2 = 1$ variable.

The linear equations are $x_1 = 0$

$x_2 + x_3 = 0$

choose $x_3 = k_1$

$x_2 = -x_3 = -k_1$.

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ where } k_1 \neq 0.$$

$x_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corrosponding to the eigen value $\lambda = -i$.

<u>case (ii) :- Eigen vector corrosponding to the eigen value $\lambda = i$</u> :-

For $\lambda = i$, The system (1) can be written as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now we reduce the co efficient matrix to echelon form by applying elementary row operations only and determine the rank of the matrix.

$$R_3 \longrightarrow R_3 + R_2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the rank of the co efficient matrix of the system is $r = 1 =$ The No. of non zero rows.

So that the system have $n - r = 3 - 1 = 2$ linearly independent solutions. There are only two linearly independent eigen vectors corrosponding to the eigen value $\lambda = i$

To determine this, we have to assign an arbitrary value for $n - r = 3 - 1 = 2$ variables.

The linear equation is $x_2 - x_3 = 0$.

choose $x_3 = k_2$

$x_2 = x_3 = k_2$.

Now we can not find x_1 from these equations. As x_1 is not present in any of these equations it follows that x_1 is an arbitrary.

Hence $x_1 = k_3$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_3 \\ k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly independent eigen vectors corrosponding to the eigen value $\lambda = i$

$\therefore$ $X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are eigen vectors corrosponding to the eigen values $\lambda = -i, i, i$.

Find the eigen values and eigen vectors of the Hermitian matrix.

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$$A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$$

Sol: Given that $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$ i.e $\begin{vmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{vmatrix} = 0$.

$$(2-\lambda)^2 - (3+4i)(3-4i) = 0.$$

$$(2-\lambda)^2 - 25 = 0$$

$$\lambda^2 - 4\lambda - 21 = 0.$$

$$\lambda^2 - 7\lambda + 3\lambda - 21 = 0$$

$$\lambda(\lambda-7) + 3(\lambda-7) = 0$$

$$(\lambda-7)(\lambda+3) = 0$$

$$\lambda = -3, 7.$$

The eigen values of the matrix A are $\lambda = -3, 7$.

Now we have find the eigen vectors $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponding to the eigen values of λ by solving the homogeneous system $(A-\lambda I)X = 0$.

$$\text{i.e } \begin{bmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \text{—(1)}.$$

Case (i) :- Eigen vector corresponding to the eigen value $\lambda = 7$.

For $\lambda = 7$ the system (1) can be written as

$$\begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now we reduce the coefficient matrix to echelon form by applying E-row operations only and determine the rank of the matrix

$R_2 \to 5R_2 + (3-4i)R_1$

$$\begin{bmatrix} -5 & 3+4i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here the rank of the co efficient matrix of the system is $r=1$.

So that the system have $n-r = 2-1 = 1$ linearly independent solutions.

$\therefore$ There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = 7$.

To determine this, we have to assign an arbitrary value for $n-r = 2-1 = 1$ variable.

The linear eqn is, $-5x_1 + (3+4i)x_2 = 0$.

choose $x_2 = k_1$

$$x_1 = \frac{3+4i}{5} x_2$$

$$x_1 = \frac{3+4i}{5} k_1.$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3+4i}{5} k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} \frac{3+4i}{5} \\ 1 \end{bmatrix} \text{ where } k_1 \neq 0.$$

$X_1 = \begin{bmatrix} \frac{3+4i}{5} \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$.

Case(ii):- Eigen vector corresponding to the eigen value $\lambda = -3$:-

For $\lambda = -3$, The system ① can be written as.

$$\begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

. Now we reduce the co efficient matrix, to echelon form by applying elementary row operations only and determine the rank of the matrix.

$$R_2 \longrightarrow 5R_2 - (3-4i)R_1$$

$$\begin{bmatrix} 5 & 3+4i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here the rank of the co efficient matrix of the system is $r=1$ so that the system have $n-r = 2-1 = 1$ linearly independent solution.

There is only one linearly independent eigen vector corrosponding to the eigen value $\lambda = -3$

To determine this, we have to assign an arbitrary value for $n-r = 2-1 = 1$ variable.

The linear eqn is $5x_1 + (3+4i)x_2 = 0$

choose $x_2 = k_2$

$$5x_1 = -(3+4i)x_2$$

$$x_1 = -\frac{(3+4i)}{5} k_2$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{(3+4i)}{5}k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} -\frac{(3+4i)}{5} \\ 1 \end{bmatrix} \quad \text{where } k_2 \neq 0.$$

$X_2 = \begin{bmatrix} -\frac{(3+4i)}{5} \\ 1 \end{bmatrix}$ is the L.I eigen vector corrosponding to the eigen value $\lambda = -3$.

$\therefore X_1 = \begin{bmatrix} \frac{3+4i}{5} \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} -\frac{(3+4i)}{5} \\ 1 \end{bmatrix}$ are the eigen vectors corrosponding to the eigen values $\lambda = 7, -3$.

Find the eigen values and corrosponding eigen vectors of the matrix 21

$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$.

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sol:- Given that $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$

$$\text{i.e} \begin{vmatrix} \frac{1}{\sqrt{3}} - \lambda & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} - \lambda \end{vmatrix} = 0.$$

$$-\left(\frac{1}{3} - \lambda^2\right) - \frac{(1+i)(1-i)}{\sqrt{3}\sqrt{3}} = 0$$

$$\lambda^2 - \frac{1}{3} - \frac{1}{3}(1+1) = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1.$$

The eigen values of the matrix A are $\lambda = 1, -1$.

Now we have to find eigen vectors $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corrosponding to eigen values of λ by solving the homogeneous System $(A-\lambda I)X = 0$

$$\text{i.e} \begin{bmatrix} \frac{1}{\sqrt{3}} - \lambda & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} - \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow (1).$$

Case(i):- Eigen vector corresponding to the eigen value $\lambda = 1$

For $\lambda = 1$, The system ① can be written as.

$$\begin{bmatrix} \frac{1}{\sqrt{3}} - 1 & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to \left(\frac{1-\sqrt{3}}{\sqrt{3}}\right) R_2 - \left(\frac{1-i}{\sqrt{3}}\right) R_1.$$

$$\begin{bmatrix} \frac{1-\sqrt{3}}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here the rank of the coefficient matrix of the system is

$r = 1 =$ No. of non zero rows.

So that the system have $n - r = 2 - 1 = 1$ linearly independent solutions

$\therefore$ There is only one linearly indepent eigen vector corresponding to the eigen value $\lambda = 1$.

To determine this, we have to assign an arbitrary value for $n - r = 2 - 1 = 1$ variable.

The linear equation is $\left(\frac{1-\sqrt{3}}{\sqrt{3}}\right) x_1 + \left(\frac{1+i}{\sqrt{3}}\right) x_2 = 0$

choose $x_2 = k_1$

$$\frac{1-\sqrt{3}}{\sqrt{3}} x_1 = \frac{-(1+i)}{\sqrt{3}} x_2$$

$$x_1 = -\frac{(1+i)}{1-\sqrt{3}} k_1.$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1+i}{\sqrt{3}-1} k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} \frac{1+i}{\sqrt{3}-1} \\ 1 \end{bmatrix}$$ where $k_1 \neq 0$.

$x_1 = \begin{bmatrix} \frac{1+i}{\sqrt{3}-1} \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$. 22 11X

case (ii):- Eigen vector corresponding to the eigen value $\lambda = -1$

For $\lambda = -1$, The system (1) can be written as.

$$\begin{bmatrix} \frac{1}{\sqrt{3}}+1 & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{-1}{\sqrt{3}}+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1+\sqrt{3}}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{-1+\sqrt{3}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by apply-ing elementary row operations only and determine the rank of the matrix.

$$R_2 \longrightarrow \left(\frac{1+\sqrt{3}}{\sqrt{3}}\right) R_2 - \left(\frac{1-i}{\sqrt{3}}\right) R_1$$

$$\begin{bmatrix} \frac{1+\sqrt{3}}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Here the rank of the coefficient matrix of the system is $r = 1 =$ No. of non zero rows.

So that the system have $n - r = 2 - 1 = 1$ linearly independent solution.

$\therefore$ There is only one linearly independent eigen vector correspon-ding to the eigen value $\lambda = -1$.

To determine this, we have to assign an arbitrary value for $n-r = 2-1 = 1$ variable

The linear equation is

$$\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)x_1 + \left(\frac{1+i}{\sqrt{3}}\right)x_2 = 0$$

choose $x_2 = k_2$

$$(1+\sqrt{3})\,x_1 = -(1+i)\,x_2$$

$$x_1 = \frac{-(1+i)}{1+\sqrt{3}}k_2$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{(1+i)}{1+\sqrt{3}}k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} -\frac{(1+i)}{1+\sqrt{3}} \\ 1 \end{bmatrix} \text{ where } k_2 \neq 0.$$

$X_2 = \begin{bmatrix} -\frac{(1+i)}{1+\sqrt{3}} \\ 1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = -1$.

$\therefore X_1 = \begin{bmatrix} \frac{1+i}{\sqrt{3}-1} \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} -\frac{(1+i)}{\sqrt{3}+1} \\ 1 \end{bmatrix}$ are two linearly independent eigen vectors corresponding to the eigen values $\lambda = 1, -1$.

Determine the constants p, q, r, s, t, u so that $[1\ 1\ 1]^T$, $[1, 0\ -1]^T$ and $[1\ -1\ 0]^T$ are the eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix}$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the eigen vector corresponding to λ_1.

$$\therefore \quad AX_1 = \lambda_1 X_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 \end{bmatrix}$$

$1+1+1 = \lambda_1$ i.e $\lambda_1 = 3$.

$p+q+r = \lambda_1 \implies p+q+r = 3$. —— (1)

$s+t+u = \lambda_1 \implies s+t+u = 3$ —— (2)

Let $x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ be the eigen vector corresponding to λ_2 Then

$$AX_2 = \lambda_2 X_2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ 0 \\ -\lambda_2 \end{bmatrix}$$

$1 \cdot 1 + 1.0 + 1(-1) = \lambda_2 \implies \lambda_2 = 0$.

$p + q.0 + r(-1) = 0 \implies p - r = 0$. —— (3)

$s + t \cdot 0 - u = -\lambda_2 \implies s - u = 0$ —— (4).

Let $x_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ be the eigen vector corresponding to λ_3 Then

$$AX_3 = \lambda_3 X_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \lambda_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_3 \\ -\lambda_3 \\ 0 \end{bmatrix}$$

$\lambda_3 = 0$.

$p - q = -\lambda_3 \Rightarrow p - q = 0$. —— ⑤

$s - t = 0$ —— ⑥.

To get the values of p, q, r, s, t, u we have to solve the equations ① to ⑥.

① + ③ + ⑤ $\Rightarrow 3p = 3 \Rightarrow p = 1$.

$\therefore$ ③ $\Rightarrow r = 1$ and ⑤ $\Rightarrow q = 1$.

Similarly from ②, ④ and ⑥ we get $s = t = u = 1$.

$\therefore p = q = r = s = t = u = 1$.

$\therefore$ The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

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Let a 3x3 matrix A have eigen values 1, 2, -1. Find the trace of the matrix $B = A - A^{-1} + A^2$. Also find determinant of B.

Sol:- Given that the eigen values of the matrix A are $\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = -1$.

We know that If λ is an eigen value of the matrix A then. $f(\lambda)$ is an eigen value of the matrix $f(A)$.

Let $f(A) = A - A^{-1} + A^2$.

An eigen values of the matrix A^2 are $\lambda_1 = 1$ $\lambda_2 = 4$ $\lambda_3 = 1$

An eigen values of the matrix A^{-1} are $\lambda_1 = 1$ $\lambda_2 = \frac{1}{2}$ $\lambda_3 = -1$.

Let $f(\lambda) = \lambda - \lambda^{-1} + \lambda^2$.

$\therefore f(\lambda_1) = f(1) = 1 - 1 + 1 = 1.$

$f(\lambda_2) = f(2) = 2 - \frac{1}{2} + 4 = \frac{11}{2}$.

$f(\lambda_3) = f(-1) = -1 - (-1) + 1 = 1.$

$\therefore$ The eigen values of the matrix $f(A)$ i.e B are $1, \frac{11}{2}$ and 1.

$\therefore$ The trace of the matrix $B = 1 + \frac{11}{2} + 1 = \frac{15}{2}$.

The determinant of the matrix $B = 1 \cdot \frac{11}{2} \cdot 1 = \frac{11}{2}$.

EIGEN VALUES AND EIGEN VECTORS

1) Find the Eigen values and Eigen vectors of a matrix A and A^3.

Where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(3)

Ans:- Eigen values of A are $\lambda = -2, 3, 6$; Eigen values of A^3 are $\lambda = -8, 27, 196$

Eigen vectors $x_1 = [1, 0, -1]^T$ $x_2 = [1, -1, 1]^T$ $x_3 = [1, 2, 1]^T$

2) Determine the Eigen values and Eigen vectors of A and A^{-1}

Where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Ans:- Eigen values of A are $\lambda = 1, 2, 3$; Eigen values of A^{-1} are $\lambda = 1, \frac{1}{2}, \frac{1}{3}$.

Eigen vectors $x_1 = [-1, 1, 0]^T$ $x_2 = [-2, 1, 2]^T$ $x_3 = [-1, 1, 2]^T$.

3) Determine the Eigen values and Eigen vectors of A and Adj A.

Where $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Ans:- Eigen values of A are $\lambda = 1, 2, -2$; Eigen values of Adj A are $\lambda = -4, -2, 2$.

Eigen vectors $x_1 = [-1, 1, 1]^T$ $x_2 = [0, 1, 1]^T$ $x_3 = [8, -5, 3]^T$.

4) Find the Eigen values and Eigen vectors a matrix A and 2A, 30A, 44A.

Where $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Ans:- Eigen values of (i) A are $\lambda = 1, 2, -2$ (ii) 2A are 2, 4, −4

(iii) 30A are 30, 60, −60 (iv) 44, 88, −88.

Eigen vectors are $x_1 = [1, 0, 0]^T$ $x_2 = [2, 1, 0]^T$ $x_3 = [-\frac{4}{3}, 1, -2]^T$

5 If $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ find the eigen values and eigen vectors of A and those of

$B = 2A^2 - \frac{1}{2}A + 3I$. Ans:- Eigen values of A are $\lambda = 4, 6$

Eigen values of B are $\lambda = 33, 72$.

Eigen vectors are $x_1 = [1, 1]^T$ $x_2 = [2, 1]^T$

6 For the matrix $A=\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}$ find the eigen values and eigen vectors of the matrix $B=2A^3-3A^2+4A-5I$. 2

Ans:- Eigen values of A are $\lambda=1,2,3$ Eigen values of B are $\lambda=-2,7,34$

Eigen vectors are $x_1=[1,0,-1]^T$ $x_2=[0,1,0]^T$ $x_3=[1,0,1]^T$

7 Let a 4x4 matrix A have eigen values $1,-1,2,-2$. Find the value of the determinant of the matrix $B=2A+A^{-1}-I$.

Ans:- $\lambda=2,-4,\frac{7}{2},-\frac{11}{2}$, $|B|=154$.

8 Let a 3x3 matrix A have eigen values $1,2,-1$. Find the trace of the matrix $B=A-A^{-1}+A^2$.

Ans:- $1,\frac{11}{2},1$, Trace of $B=\frac{15}{2}$.

Matrix Polynomial :—

An expression of the form $F(x) = A_0 + A_1x + A_2x^2 + \dots + A_mx^m$, $A_m \neq 0$.
Where $A_0, A_1, A_2, \dots A_m$ are matrices each of order $n\times n$ over a field F, is called a matrix polynomial of degree m.

The symbol x is called indeterminate and will be assumed that it is commutative with every matrix coefficient.

The matrices themselves are matrix polynomials of zero degree.

Equality of Matrix Polynomials :—

Two matrix polynomials are equal if and only if the coefficients of like powers of x are the same.

The Cayley Hamilton Theorem :—

Every square matrix satisfies its own characteristic equation.

Determination of A^{-1} using Cayley Hamilton Theorem :—

The matrix A satisfies its characteristic equation.

i.e $(-1)^n \left[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I\right] = 0$

$$\Rightarrow A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0.$$

Multiplying bothsides by A^{-1}, we get.

$$A^{-1}\left[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I\right] = 0$$

$$A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \dots + a_n A^{-1} = 0$$

If A is non singular, then we have.

$$a_n A^{-1} = -A^{n-1} - a_1 A^{n-2} - a_2 A^{n-3} - \dots - a_{n-1} I.$$

$$A^{-1} = \frac{-1}{a_n}\left[A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \dots + a_{n-1} I\right].$$

11) Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by using Cayley Hamilton theorem.

verify Cayley Hamilton theorem and hence find A^4.

Sol:- Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$.

i.e $\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$.

$$(1-\lambda)\left[(1-\lambda)(2-\lambda)-1\right] + 1\left[0-2\right] = 0.$$

$$(1-\lambda)\left[2-3\lambda+\lambda^2-1\right]-2 = 0$$

$$(1-\lambda)\left[\lambda^2-3\lambda+1\right]-2 = 0$$

$$\lambda^2-3\lambda+1-\lambda^3+3\lambda^2-\lambda-2 = 0$$

$$-\lambda^3+4\lambda^2-4\lambda-1 = 0$$

$$\lambda^3-4\lambda^2+4\lambda+1 = 0.$$

We know that the Cayley hamilton theorem.

Every square matrix satisfies its own characteristic equation.

$$A^3-4A^2+4A+I = 0.$$

Multiply bothsides by A^{-1}, we get

$$A^{-1}(A^3-4A^2+4A+I) = A^{-1}(0)$$

$$A^2-4A+4I+A^{-1} = 0.$$

$$A^{-1} = -A^2+4A-4I.$$

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$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -2 & -2 & -3 \\ -6 & -1 & -5 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 1 \\ -2 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}.$$

Veribication :-

We know that the cayley Hamilton theorem.

Every square matrix satisfies its own characteristic equation.

i.e $A^3 - 4A^2 + 4A + I = 0$.

$$A^3 = A^2 . A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix}$$

$$A^3 - 4A^2 + 4A + I = \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - 4\begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} + 4\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3 - 4A^2 + 4A + I = 0$

$\therefore$ cayley Hamilton theorem is veribied.

To find A^4 :-

We have $A^3 - 4A^2 + 4A + I = 0$

Pre multiply with 'A', we get

$$A(A^3 - 4A^2 + 4A + I) = A(0).$$

$$A^4 - 4A^3 + 4A^2 + A = 0$$

$$A^4 = 4A^3 - 4A^2 - A$$

$$A^4 = 4\begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - 4\begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -9 & -7 & -12 \\ 24 & 3 & 19 \\ 38 & -5 & 22 \end{bmatrix}.$$

Using Cayley-Hamilton theorem, Express $A^6-4A^5+8A^4-12A^3+14A^2$ as a linear polynomial in A, where $A=\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}$

sol:- Given that $A=\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

i.e $\begin{vmatrix}1-\lambda & 2\\ -1 & 3-\lambda\end{vmatrix}=0$

$(1-\lambda)(3-\lambda)+2=0.$

$\lambda^2-4\lambda+5=0.$

We know that The Cayley Hamilton Theorem.

Every square matrix satisfies its own characteristic equation

i.e $A^2-4A+5I=0.$

$A^2=4A-5I.$ —— (1)

Pre multiplying (1) by A, A^2, A^3 and A^4, we get

$A^3=4A^2-5A.$

$A^4=4A^3-5A^2$

$A^5=4A^4-5A^3$

$A^6=4A^5-5A^4.$

$A^6-4A^5+8A^4-12A^3+14A^2=4A^5-5A^4-4A^5+8A^4-12A^3+14A^2$

$=3A^4-12A^3+14A^2$

$=3(4A^3-5A^2)-12A^3+14A^2$

$=12A^3-15A^2-12A^3+14A^2$

$=-A^2$

$=5I-4A.$

Which is a linear polynomial in A.

Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find A^4 and A^{-1}.

Hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

sol: Given that $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$$

$$(1-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$$

Verification:-

We know that cayley Hamilton theorem.

Every square matrix satisfies its own characteristic equation.

$$\text{i.e } A^3 - 5A^2 + 7A - 3I = 0.$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$A^3-5A^2+7A-3I = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3-5A^2+7A-3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

$\therefore$ Cayley Hamilton theorem verified.

<u>To find A^4:-</u>

We have $A^3-5A^2+7A-3I = 0$.

Pre multiply with A, we get

$$A(A^3-5A^2+7A-3I) = A(0)$$

$$A^4-5A^3+7A^2-3A = 0$$

$$A^4 = 5A^3-7A^2+3A$$

$$A^4 = 5\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 7\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 3\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

$$A^4 = \begin{bmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{bmatrix}$$

To find A^{-1} :-

We have $A^3 - 5A^2 + 7A - 3I = 0$.

Pre multiply with A^{-1}, we get

$$A^{-1}(A^3 - 5A^2 + 7A - 3I) = A^{-1}(0)$$

$$A^2 - 5A + 7I - 3A^{-1} = 0.$$

$$3A^{-1} = A^2 - 5A + 7I.$$

$$3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

→ We have $A^3 - 5A^2 + 7A - 3I = 0$..

$$A^3 = 5A^2 - 7A + 3I.$$

Pre multiply with 'A', we get

$$A^4 = 5A^3 - 7A^2 + 3A.$$

$$A^5 = 5A^4 - 7A^3 + 3A^2$$

$$A^6 = 5A^5 - 7A^4 + 3A^3$$

$$A^7 = 5A^6 - 7A^5 + 3A^4$$

$$A^8 = 5A^7 - 7A^6 + 3A^5.$$

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$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

$$= (5A^7 - 7A^6 + 3A^5) - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^4 - 5A^3 + 8A^2 - 2A + I.$$

$$= (5A^3 - 7A^2 + 3A) - 5A^3 + 8A^2 - 2A + I.$$

$$= A^2 + A + I.$$

$$\therefore A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

If $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$ then show that $A^n = A^{n-2}+A^2-I$ for $n\geq 3$. Hence find A^{50}. (31)

Sol:- Given that $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

$$\text{i.e} \begin{vmatrix}1-\lambda & 0 & 0\\ 1 & 0-\lambda & 1\\ 0 & 1 & 0-\lambda\end{vmatrix}=0$$

$$(1-\lambda)(\lambda^2-1)=0.$$

$$\lambda^3-\lambda^2-\lambda+1=0.$$

We know that Cayley Hamilton theorem.

Every square matrix satisfies its own characteristic equation.

$$A^3-A^2-A+I=0.$$

$$A^3-A^2=A-I.$$

Pre multiplying both sides successively by A, we obtain.

$$A^3-A^2=A-I.$$

$$A^4-A^3=A^2-A$$

$$A^5-A^4=A^3-A^2$$

$$\cdots\cdots\cdots$$

$$A^{n-1}-A^{n-2}=A^{n-3}-A^{n-4}$$

$$A^n-A^{n-1}=A^{n-2}-A^{n-3}$$

Adding these equations, we get

$$A^n-A^2=A^{n-2}-I.$$

$$A^n=A^{n-2}+A^2-I,\quad n\geq 3.$$

Using this equation recursively, we get.

$$A^{n-2}=A^{(n-2)-2}+A^2-I=A^{n-4}+A^2-I.$$

$$A^n=(A^{n-4}+A^2-I)+A^2-I.$$

$$A^n=A^{n-4}+2(A^2-I)$$

$$= (A^{n-6} + A^2 - I) + 2(A^2 - I) = A^{n-6} + 3(A^2 - I)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$= A^{n-(n-2)} + \frac{1}{2}(n-2)\cdot(A^2 - I).$$

$$= \frac{n}{2}A^2 - \frac{1}{2}(n-2)I.$$

Substituting $n=50$, we get

$$A^{50} = 25A^2 - 24I = 25\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} - 24\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\therefore A^{50} = \begin{bmatrix}1 & 0 & 0\\ 25 & 1 & 0\\ 25 & 0 & 1\end{bmatrix}.$$

CAYLEY- HAMILTON THEOREM

1 State Cayley Hamilton theorem

2 Verify Cayley Hamilton theorem for the matrix $A=\begin{bmatrix}1 & 2 & 0\\ -1 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}$

Hence find (i) A^{-1} (ii) A^4.

Ans:- $A^{-1}=\frac{1}{3}\begin{bmatrix}-3 & -2 & 4\\ 3 & 1 & -2\\ -3 & 0 & 3\end{bmatrix}$.

3 Verify Cayley Hamilton theorem for the matrix $A=\begin{bmatrix}0 & 2 & -1\\ -2 & 0 & 2\\ 1 & -2 & 0\end{bmatrix}$

Hence find (i) A^4 and show that (ii) $A^3=-9A$ (iii) $A^5=81A$.

4 Prove that the matrix $A=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$ satisfies its characteristic equation

Using C.H.T. show that (i) $A^4=I$ and (ii) $A^3=A^{-1}$. Also Find A^4.

5 If $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$ find A^8 using the cayley Hamilton theorem.

6 Verify Cayley Hamilton theorem for the matrix $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$.

Express $A^4-3A^3+2A^2-5I$ as a linear polynomial in A.

7 If $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$. Verify Cayley Hamilton theorem for the matrix A.

Hence find (i) A^4 (ii) A^{-1}. Also Find the matrix

$B=A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$ Ans: A^2+A+I

8 If $A=\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}$ show that $A^8-4A^5+8A^4-12A^3+14A^2=\begin{bmatrix}1 & -8\\ 4 & -7\end{bmatrix}$.

9 For the matrix $A=\begin{bmatrix}8 & -8 & 2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}$ Express A^3, A^4 and A^{-1} in terms of I, A and A^2

by using the cayley Hamilton Theorem. Hence find these explicitely.

Ans:- $A^3=\begin{bmatrix}214 & -296 & 206\\ 88 & -115 & 70\\ 69 & -100 & 69\end{bmatrix}$ $A^4=\begin{bmatrix}1146 & -1904 & 1226\\ 322 & -639 & 476\\ 359 & -544 & 407\end{bmatrix}$ $A^{-1}=\begin{bmatrix}9 & 0 & -22\\ 10 & -4 & -24\\ 7 & -8 & -10\end{bmatrix}\frac{1}{22}$

10 If $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ find A^3, A^4 and A^{-2} by using Cayley Hamilton theorem.

Ans:- $A^3 = \begin{bmatrix} 135 & 152 \\ 140 & 163 \\ 60 & 76 \end{bmatrix}$ $A^4 = \begin{bmatrix} 935 & 1173 & 1633 \\ 1000 & 1162 & 1677 \\ 475 & 554 & 759 \end{bmatrix}$ $A^{-2} = \frac{1}{245}\begin{bmatrix} -5 & -23 & 69 \\ 32 & 10 & -79 \\ -17 & 10 & 19 \end{bmatrix}$

11 Find the characteristic equation of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & 1 & 5 \end{bmatrix}$ and show that it is satisfied by A and hence obtain its A^{-1}.

Ans:- $A^{-1} = \frac{1}{72}\begin{bmatrix} 24 & -6 & -6 \\ 4 & 14 & 2 \\ -4 & 4 & 16 \end{bmatrix}$

12 Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and show that it is satisfied by A and hence obtain its A^{-1}.

Ans:- $A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

13 Using Cayley Hamilton theorem, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. Where $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ Ans:- $A + 5I$.

14 Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies the Cayley Hamilton theorem. Hence find A^4.

15 If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ show by using the Cayley-Hamilton theorem that

(i) $A^4 = 2A^2 - I$ (ii) $A^5 = 2A^2 + A - 2I$.

DIAGONALIZATION OF A MATRIX :-

Let A be a square matrix of order n. Then A is said to be diagonalizable if there exists a matrix P of order n such that $P^{-1}AP = D$ where D is a diagonal matrix. Then $P^{-1}AP$ is a diagonal form of A.

P is formed by the linearly independent eigen vectors corresponding to the eigen values of A then $P = [x_1\ x_2\ x_3 \dots x_n]$ is said to be transforming matrix of A and it reduces the matrix A to the diagonal form D.

Similarity of Matrices :-

Let A and B are square matrices of order n. Then B is said to be similar to A if there exists a non singular matrix P such that $B = P^{-1}AP$.

Algebraic and Geometric multiplicities of a characteristic root :-

If λ be a characteristic root of a order 't' of the characteristic equation $|A-\lambda I|=0$, then t is called the "algebraic multiplicity" of λ i.e the order of the characteristic λ, is said to be "algebraic multiplicty". It is denoted by "t".

If s is the number of linearly independent Eigen vectors corresponding to the Eigen value λ, then 's' is called the "geometric multiplicity" of λ i.e The number of linearly independent Eigen vectors corresponding to the Eigen value λ, is said to be its geometric multiplicity. It is denoted by 's'.

The geometric multiplicity of a characteristic root cannot exceed its algebraic multiplicity i.e $s \le t$.

Note :-

(2)

(i) If A is similar to a diagonal matrix B then the diagonal elements of B are the eigen values of A.

(ii) If A is a square matrix of order n is diagonalizable iff it possesses n linearly independent eigen vectors.

(iii) If the Eigen values of an $n \times n$ matrix are all distinct, then it is always similar to a diagonal matrix i.e a diagonalizable matrix

(iv) If the Eigen values of a matrix are not distinct, then we have to verify the following condition or test for the diagonalization of a matrix.

Condition for the diagonalization :-

The necessary and sufficient condition for a square matrix A to be diagonalizable is that the geometric multiplicity of each of its Eigen values coincides with the algebraic multiplicity.

Modal and Spectral Matrices :-

If a square matrix A is diagonalizable then the matrix P which transforms A to the diagonal form D is called the modal matrix of A and the matrix D is called the spectral matrix of A.

Let x_1, x_2, x_3 are Eigen vectors corresponding to the Eigen values $\lambda_1, \lambda_2, \lambda_3$ of A respectively then the modal matrix of A is $P = [x_1 \ x_2 \ x_3]$ and the spectral matrix of A is $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

exists such that $P^{-1}AP = D$.

Working procedure to Diagonalize a square matrix A :-

Let the square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Step 1 :- The characteristic equation of A is $|A-\lambda I| = 0$

i.e $\begin{vmatrix} a_{11}-\lambda & a_{21} & a_{31} \\ a_{21} & a_{22}-\lambda & a_{32} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$

Step 2 :- Solve the characteristic equation and find the Eigen values $\lambda_1, \lambda_2, \lambda_3$ of the given matrix A.

Step 3 :-

Case (i) :- The Eigen values of matrix A are distinct.

(a) Find the Eigen vectors X_1, X_2, X_3 corresponding to the Eigen values λ_1, λ_2 and λ_3.

Let $X_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ $X_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ $X_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$

(b) Consider the Modal Matrix $P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

(c) Find $P^{-1} = \frac{1}{|P|} \text{Adj} P$.

(d) Find $P^{-1}AP$ which is the diagonal matrix of A.

$$P^{-1}AP = D = \text{Diag}[\lambda_1 \; \lambda_2 \; \lambda_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Case (ii) :- The Eigen values of matrix A are not distinct.

Suppose $\lambda_1 = \lambda_2$ and λ_3 is distinct.

Here algebraic multiplicity of $\lambda_1 = 2$ and algebraic multiplicity of $\lambda_3 = 1$.

(a) Find the Eigen vectors corresponding to the Eigen values λ_1, λ_2 and λ_3

Let X_1, X_2 are the Eigen vectors corresponding to the Eigen value λ_1 and X_3 is the Eigen vector corresponding to the Eigen value λ_3.

Let $X_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ $X_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ $X_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$

Geometric multiplicity of $\lambda_1 = 2$, Geometric multiplicity of $\lambda_3 = 1$.

$\therefore$ Algebraic multiplicity of λ_1 = Geometric multiplicity of $\lambda_1 = 2$

Algebraic multiplicity of λ_3 = Geometric multiplicity of $\lambda_3 = 1$

(b) Consider the Modal matrix $P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

(c) Find $P^{-1} = \frac{1}{|P|} \operatorname{Adj} P$.

(d) Find $P^{-1}AP$ which is the diagonal matrix of A.

$\therefore P^{-1}AP = D = \operatorname{Diag}[\lambda_1 \; \lambda_2 \; \lambda_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Case (iii) :- The Eigen values of matrix A are not distinct.

Suppose $\lambda_1 = \lambda_2$ and λ_3 is distinct.

Here algebraic multiplicity of $\lambda_1 = 2$, algebraic multiplicity of $\lambda_3 = 1$.

(a) Find the Eigen vectors corresponding to the Eigen values $\lambda_1 . \lambda_2 . \lambda_3$.

Let X_1 be the Eigen vector corresponding to Eigen value λ_1

Let X_3 be the Eigen vector corresponding to Eigen value λ_3.

Here Algebraic multiplicity of $\lambda_1 \neq$ Geometric multiplicity of λ_1

$\therefore$ A is not diagonalizable.

Computation of positive integral powers of matrix A :-

Let A be a square matrix of order 3. Then there exists a non singular matrix P such that $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Where $\lambda_1, \lambda_2, \lambda_3$ are Eigen values of A.

$$P^{-1}AP = D$$

$$(P^{-1}AP)^2 = D^2$$

$$(P^{-1}AP)(P^{-1}AP) = D^2$$

$$P^{-1}A(PP^{-1})AP = D^2$$

$$P^{-1}AIAP = D^2$$

$$P^{-1}A^2P = D^2$$

Similarly $P^{-1}A^3P = D^3$

$$\cdots$$

$$P^{-1}A^nP = D^n \quad \text{——} \quad (1)$$

Now pre multiplying the eqn (1) with P and post multiplying with P^{-1}

we have $P(P^{-1}A^nP)P^{-1} = PD^nP^{-1}$

$$(PP^{-1})A^n(PP^{-1}) = PD^nP^{-1}$$

$$IA^nI = PD^nP^{-1}$$

$$A^n = PD^nP^{-1}$$

$\therefore A^n = PD^nP^{-1}$ Where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$

→ Find the matrix P which transforms the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to the diagonal form. Hence evaluate A^4.

Sol:- Given that $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ and λ is an eigen value of A.

The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)\left[(2-\lambda)(3-\lambda) - 2\right] - \left[2 - 2(2-\lambda)\right] = 0.$

$(1-\lambda)(6 - 5\lambda + \lambda^2 - 2) - 2\lambda + 2 = 0.$

$\lambda^2 - 5\lambda + 4 - \lambda^3 + 5\lambda^2 - 4\lambda - 2\lambda + 2 = 0.$

$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ —— (1).

$\lambda = 1$ is one of the roots of the equation (1).

$$\begin{array}{c|cccc} \lambda=1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 1, 2, 3.$

∴ Eigen values of A are $\lambda = 1, 2, 3$.

The Eigen values of A are distinct.

∴ The matrix A is diagonalizable.

Now we have to find Eigen vectors $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corresponding to Eigen value λ are obtained by solving the homogeneous system

$(A - \lambda I)X = 0$

i.e $\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — (2).

Case (i):- Eigen vector corresponding to Eigen value $\lambda = 1$:-

For $\lambda = 1$, The system (1) can be written as .

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the co eff. matrix into echelon form by applying E-row operations only and hence determine rank of coeff. matrix.

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Which is in echelon form .

Here rank of the coefficient matrix of the system is 2 i.e $r=2$

Sothat the system has $n-r = 3-2 = 1$ L.I solution .

There is only one L.I eigen vector corresponding to the Eigen value $\lambda = 1$.

To determine this we have to assign an arbitrary value for $n-r = 3-2 = 1$ variable .

From the above system, the equations can be written as

$$x_1 + x_2 + x_3 = 0$$

$$-x_3 = 0 \implies x_3 = 0 .$$

To determine this we have to assign an arbitrary value for $n-r = 3-2 = 1$ variable.

From the above system, the eqn's can be written as.

$$-x_1 - x_3 = 0 \Rightarrow x_1 + x_3 = 0$$

$$2x_2 - x_3 = 0$$

choose $x_2 = k_2$

$$x_3 = 2x_2$$

$$x_3 = 2k_2$$

$$x_1 = -x_3 = -2k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k_2 \\ k_2 \\ 2k_2 \end{bmatrix} = k_2 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \text{ where } k_2 \neq 0$$

$X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 2$.

Case (iii) Eigen vector corresponding to the eigen value $\lambda = 3$:-

For $\lambda = 3$ The system ② can be written as

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by applying E-row operations only. and hence determine the coeff. matrix.

$R_2 \to 2R_2 + R_1 \quad R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

$$x_1 + x_2 = 0 \quad [\because x_3 = 0]$$

choose $x_2 = k_1$

$$x_1 = -x_2 = -k_1.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ where } k_1 \neq 0.$$

$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$.

Case (ii) :- Eigen vector corresponding to the Eigen value $\lambda = 2$:-

For $\lambda = 2$, The system ② can be written as

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by applying E-row operations only and hence determine rank of coeff. matrix.

$$R_2 \to R_2 + R_1, \quad R_3 \to R_3 + 2R_1$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the Co.eff. matrix of the system is 2 i.e $r = 2$.

Sothat the system has $n - r = 3 - 2 = 1$ L.I solution.

There is only one L.I eigen vector corresponding to the eigen value $\lambda = 2$.

Here the rank of the coeff. matrix of the system is 2 i.e $r=2$

The system has $n-r=3-2=1$ L.I solution.

There is only one linearly independent eigen vector corresponding to the eigen value $\lambda=3$.

To determine this we have to assign an arbitrary value for $n-r=3-2=1$ variable.

From the above system, the eqn's can be written as

$$-2x_1-x_3=0 \Rightarrow 2x_1+x_3=0$$

$$-2x_2+x_3=0 \Rightarrow 2x_2-x_3=0$$

choose $x_1=k_3$ Then $x_3=-2k_3$

$2x_2=x_3$ Then $2x_2=-2k_3$

$x_2=-k_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_3 \\ -k_3 \\ 2k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$ where $k_3 \neq 0$.

$X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda=3$.

Consider $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$

Which is the modal matrix.

$$|P| = -1(-2+2)+2(-2-0)+1(2-0)$$

$$|P| = -2$$

$$P^{-1} = \frac{1}{|P|}\,\text{adj}\,P = \frac{-1}{2}\begin{bmatrix} 0 & -2 & 1 \\ 2 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ -2 & 2 & -1 \end{bmatrix}$$

Thus the matrix P transforms the matrix A to the diagonal form which is given by $P^{-1}AP = D$.

$$P^{-1}A = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -4 & -4 & 0 \\ -6 & -6 & -3 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -4 & -4 & 0 \\ -6 & -6 & -3 \end{bmatrix}\begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{Diag}(1,2,3) = D.$$

Hence $P^{-1}AP$ is a diagonal matrix.

where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is the spectral matrix.

<u>To find A^4 :-</u>

We have $A^n = PD^nP^{-1}$

$n=4$, $A^4 = PD^4P^{-1}$

$$A^4 = \frac{1}{2}\begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}.$$

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→ Diagonalize the matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$ and Hence find A^4.

Sol. Given that $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$ and λ is an eigen value of A

The characteristic eqn. of A is $|A - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{vmatrix} = 0$$

$C_2 \to C_2 - C_1, \quad C_3 \to C_3 - C_1$

$$(2-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda)^2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda)^2 = 0$$

$$\lambda = 1, 1, 2.$$

$\therefore$ Eigen values of A are $\lambda = 1, 1, 2$.

The Algebraic multiplicities of each eigen values 1 and 2 are 2 and 1.

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Now the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corresponding to the eigen value λ are obtained by solving the homogeneous system $(A-\lambda I)X = 0$

i.e $\begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — ①

Case (i) :- Eigen vector corresponding to the Eigen value $\lambda = 1$:-

For $\lambda = 1$, The system ① can be written as

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by using E-row operations only.

$R_2 \to 2R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the coeff. matrix of the system is 1 i.e $r = 1$

So that the system has $n - r = 3 - 1 = 2$ L.I solutions.

There are two linearly independent eigen vectors corresponding to the eigen value $\lambda = 1$.

To determine this we have to assign an arbitrary value for $n - r = 3 - 1 = 2$ variable.

From the above system, the eqn's can be written as

$2x_1 + 2x_2 + 2x_3 = 0$ i.e $x_1 + x_2 + x_3 = 0$

choose $x_2 = K_1$, $x_3 = K_2$

$x_1 = -x_2 - x_3 = -K_1 - K_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K_1 - K_2 \\ K_1 \\ K_2 \end{bmatrix} = K_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent eigen vectors.

Corresponding to the eigen value $\lambda = 1$.

The geometric multiplicity of the eigen value $\lambda = 1$ is 2.

Case (ii) Eigen vector Corresponding to the Eigen value $\lambda = 2$:-

For $\lambda = 2$, The system ① can be written as

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ -2 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the co eff. matrix into echelon form by applying E-row operations only and hence determine the rank of Coeff. matrix

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 + 2R_1$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the Coeff. matrix of the system is 2 i.e $r = 2$

Sothat the system has $n - r = 3 - 2 = 1$ linearly independent sol.

There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = 2$.

To determine this we have to assign an arbitrary value for $n - r = 3 - 2 = 1$ variable.

From the above system, The equations can be written as

$$x_1 + 2x_2 + 2x_3 = 0$$

$$-2x_2 - x_3 = 0 \Rightarrow 2x_2 + x_3 = 0$$

choose $x_2 = k_3$

$$x_3 = -2x_2 = -2k_3$$

$$x_1 = -2x_2 - 2x_3$$

$$x_1 = -2k_3 + 4k_3$$

$$x_1 = 2k_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_3 \\ k_3 \\ -2k_3 \end{bmatrix} = k_3 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ where } k_3 \neq 0.$$

$X_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is the L.I eigen vector corresponding to the eigen value $\lambda = 2$.

The geometric multiplicity of the eigen value $\lambda = 2$ is 1.

Since the geometric multiplicity of each eigen value of A coincides with the algebraic multiplicity.

$\therefore$ A is diagonalizable matrix.

The modal matrix $P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

$$|P| = \begin{vmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{vmatrix}$$

$$|P| = -1(0-1) + 1(-2-0) + 2(1-0)$$

$$|P| = 1$$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P.$$

Cofactor matrix of $P = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 2 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

$$\operatorname{Adj} P = [\text{Cofactor matrix of } P]^T = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus the matrix P transforms the matrix A to the diagonal form which is given by $P^{-1}AP = D$

$$P^{-1}A = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D = \text{Diag}(1, 1, 2)$$

Hence $P^{-1}AP$ is diagonal matrix.

Where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is the spectral matrix.

<u>To find A^4 :-</u>

We have $A^n = P D^n P^{-1}$

$n=4$, $A^4 = P D^4 P^{-1}$

$$A^4 = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 32 \\ 1 & 0 & 16 \\ 0 & 1 & -32 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 32 & 30 & 30 \\ 15 & 16 & 15 \\ -30 & -30 & -29 \end{bmatrix}$$

Find an orthogonal matrix that will diagonalize the real symmetric matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Also find the resulting diagonal matrix.

Sol:- Given that $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$.

$$\text{i.e} \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$R_2 \to R_2 + R_3$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 2-\lambda & 2-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$C_3 \to C_3 - C_2$

$$(2-\lambda)\begin{vmatrix} 6-\lambda & -2 & 4 \\ 0 & 1 & 0 \\ 2 & -1 & 4-\lambda \end{vmatrix} = 0.$$

Expanding by R_2, we have.

$$(2-\lambda)\left[(6-\lambda)(4-\lambda)-8\right] = 0$$

$$(2-\lambda)(\lambda^2-10\lambda+16) = 0.$$

$$(2-\lambda)(\lambda-2)(\lambda-8) = 0$$

$$\lambda = 2, 2, 8.$$

$\therefore$ The Eigen values of A are 2, 2, 8 which are not distinct.

The algebraic multiplicities of each eigen values 2 and 8 are 2 and 1.

Now the Eigen vector corresponding to the Eigen values λ are obtained by solving the system of equations $(A-\lambda I)X = 0$ —— ①.

Case(i):- Eigen vector corresponding to the Eigen value $\lambda = 2$:-

For $\lambda = 2$ The system ① can be written as

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the coeff. matrix into echelon form by applying E-row operations only. $R_2 \to 2R_2 + R_1 \quad R_3 \to 2R_3 - R_1$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Rank of the co efficient matrix of the system is 2 i.e $r=2$.

There is so that the homogeneous system has $n-r = 3-2=1$ linearly independent solution. corresponding to the eigen value $\lambda = 2$.

There is only one linearly independent eigen vector.

To determine this ; From the above system

The equations can be written as .

$$4x_1 - 2x_2 + 2x_3 = 0 .$$

$$2x_1 - x_2 + x_3 = 0 .$$

choose $x_1 = k_1 \quad x_2 = k_2$

$$x_3 = x_2 - 2x_1$$

$$x_3 = k_2 - 2k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_2 - 2k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are two linearly independent eigen vectors 62

corresponding to the eigen value $\lambda = 2$.

So that the geometric multiplicity of the eigen value $\lambda = 2$ is 2.

Case (iii):- Eigen Vector corresponding to the Eigen value $\lambda = 8$.

For $\lambda = 8$, The system ① can be written as.

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the Coeff. matrix into echelon form by applying E-row operations only.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$ = The No. of Non zero rows equivalent to A.

$\rho(A) = 2 < 3$ (No. of non zero rows)

So that the homogeneous system have $n - r = 3 - 2 = 1$ linearly independent solutions.

There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = 8$.

To determine this, we have to assign an arbitrary value for one variable.

From the above system the linear equations are

$$x_1 + x_2 - x_3 = 0$$
$$x_2 + x_3 = 0$$

choose $x_3 = k_3$

$$x_2 = -x_3 = -k_3$$

$$x_1 = x_3 - x_2 = +k_3 + k_3 = 2k_3$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_3 \\ -k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 8$.

So that the geometric multiplicity of $\lambda = 8$ is 1.

Since the geometric multiplicity of each eigen value of A coincides with the algebraic multiplicity

$\therefore$ A is diagonalizable matrix.

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ are the eigen vectors corresponding to the eigen values $\lambda = 2, 2, 8$.

Here the eigen vectors X_1 and X_2 are not pairwise orthogonal.

Now we have to find the anothere linearly independent eigen vector X_2 pairwise orthogonal to X_1 and X_3.

Let $X_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the another linearly independent eigenvector corresponding to $\lambda = 2$ and is orthogonal to X_1 and X_3.

x_1, x_2 are pairwise orthogonal if $a + 0.b - 2c = 0$.

x_2, x_3 are pairwise orthogonal if $2a - b + c = 0$.

Solving above two equations, we get

$$\frac{a}{-2} = \frac{b}{-5} = \frac{c}{-1} \qquad \begin{matrix} 0 & -2 & 1 & 0 \\ -1 & 1 & 2 & -1 \end{matrix}$$

$$a = -2 \quad b = -5 \quad c = -1$$

$\therefore$ The Eigen vectors $x_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $x_2 = \begin{bmatrix} -2 \\ -5 \\ -1 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ are pairwise orthogonal.

Consider the modal matrix $[x_1 \ x_2 \ x_3] = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -5 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

$$\|x_1\| = \sqrt{1+0+4} = \sqrt{5} \qquad \|x_2\| = \sqrt{4+25+1} = \sqrt{30}$$

$$\|x_3\| = \sqrt{4+1+1} = \sqrt{6}.$$

Normalized modal matrix. $P = \left[\frac{x_1}{\|x_1\|} \ \frac{x_2}{\|x_2\|} \ \frac{x_3}{\|x_3\|}\right]$

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{-5}{\sqrt{30}} & \frac{-1}{\sqrt{6}} \\ \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Which is the orthogonal matrix.

By definition $PP^T = P^TP = I \implies P^{-1} = P^T$

The matrix P will reduce the matrix A to the diagonal form which is given by $P^{-1}AP = D$ i.e $P^TAP = D$.

$$P^T A P = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \\ \frac{-2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} & \frac{-1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{-5}{\sqrt{30}} & \frac{-1}{\sqrt{6}} \\ \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = D$$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ is the spectral matrix.

DIAGONALIZATION OF A MATRIX.

1 Define Modal matrix

2 Define Spectral matrix

3 Define Similarity of matrices.

4 Explain Diagonalization of a square matrix.

2

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1 Show that the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ is diagonalizable. Hence find P such that $P^{-1}AP$ is a diagonal matrix. Then, obtain the matrix $B = A^2+5A+3I$.

Ans:- $\lambda = 1, 2, 3$; $x_1 = [1 \ -1 \ 1]^T$, $x_2 = [1 \ 0 \ 1]^T$, $x_3 = [0 \ 1 \ 1]^T$

$$P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 25 & 8 & -8 \\ -18 & 9 & 18 \\ -2 & 8 & 19 \end{bmatrix}$$

2 Show that the matrix $A = \begin{bmatrix} -3 & -2 & 1 \\ -2 & 0 & 4 \\ -6 & -3 & 5 \end{bmatrix}$ is diagonalizable. Find the matrix P such that $P^{-1}AP$ is a diagonal matrix.

Ans:- $\lambda = 1, x_1 = [1, -2, 0]^T$; $\lambda = -1, x_2 = [3, -2, 2]^T$; $\lambda = 2, x_3 = [-1, 3, 1]^T$

$$P = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \quad P^{-1} = \frac{1}{2}\begin{bmatrix} -8 & -5 & 7 \\ 2 & 1 & -1 \\ 4 & -2 & 4 \end{bmatrix}$$

3 Show that the matrix $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$ is diagonalizable. Find the matrix P such that $P^{-1}AP$ is a diagonal matrix.

Ans:- $\lambda = 0, x_1 = [3, 1, -2]^T$; $\lambda = 2i, x_2 = [3+i, 1+3i, -4]^T$; $\lambda = -2i$, $x_3 = [3-i, 1-3i, -4]^T$

$$P = \begin{bmatrix} 3 & 3+i & 3-i \\ 1 & 1+3i & 1-3i \\ -2 & -4 & -4 \end{bmatrix} \quad P^{-1} = \frac{1}{32}\begin{bmatrix} 24 & -8 & 16 \\ 2i-6 & 2-6i & -8 \\ -2i-6 & 2+6i & 8 \end{bmatrix}.$$

R.NO	Q.NO
C1 – F0	1, 3, 5, 7, 9, 11, 13, 15,
F1 – J0	2, 4, 6, 8, 10, 12, 14, 16.

4 Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ Hence determine A^4. 6

Ans:- $\lambda = 0,\ x_1 = [1, 0, -1]^T;\ \lambda = 1,\ x_2 = [-1, -1, 1]^T;\ \lambda = 2,\ x_3 = [1, 1, 0]^T$

$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

5 Diagonalize the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ Hence determine A^3

Ans:- $\lambda = 1,\ x_1 = [1, -1, -1]^T;\ \lambda = 2,\ x_2 = [0, 1, 1]^T;\ \lambda = -2;\ x_3 = [8, -5, 7]^T$

$P = \begin{bmatrix} 1 & 0 & 8 \\ -1 & 1 & -5 \\ -1 & 1 & 7 \end{bmatrix}$ $P^{-1} = \frac{1}{12}\begin{bmatrix} 12 & 8 & -8 \\ 12 & 15 & -3 \\ 0 & -1 & 1 \end{bmatrix}$

6 Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ Hence determine A^5.

Ans:- $\lambda = 1,\ x_1 = [+1, +2, -2]^T;\ \lambda = 2,\ x_2 = [1, 1, 0]^T;\ \lambda = 3,\ x_3 = [1, 1, 1]^T$

$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$ $A^5 = \begin{bmatrix} -359 & 391 & 211 \\ -360 & 392 & 211 \\ -484 & 484 & 243 \end{bmatrix}$

7 Diagonalize the matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ Hence determine A^4.

Ans:- $\lambda = 2, 2,\ x_1 = [1, 0, -1]^T\ x_2 = [-2, 1, 0]^T\ \lambda = 4,\ x_3 = [1, 0, 1]^T$

$P = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}\frac{1}{2}$

8 Diagonalize the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ Hence determine A^6

Ans:- $\lambda = 1,\ x_1 = [3, -1, 3]^T;\ \lambda = 2, 2\ \ x_2 = [2, 0, 1]^T\ x_3 = [2, 1, 0]^T$

$P = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & -5 \\ -1 & 3 & 2 \end{bmatrix}$

9 Diagonalize the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ Hence determine A^3.

$\lambda = 1, 1 \quad x_1 = [1, 0, -1]^T \; x_2 = [0, 1, -2]^T; \quad \lambda = 5, \; x_3 = [1, 1, 1]^T$

$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

10 Diagonalize the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ Hence determine A^4

Ans:- $\lambda = 5, \quad x_1 = [1, 2, -1]^T; \; \lambda = -3, -3, \quad x_2 = [-2, 1, 0]^T \; x_3 = [3, 0, 1]$

$P = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

11 Diagonalize the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ Hence determine A^5

Ans:- $\lambda = 2, 2. \quad x_1 = [1, 2, 0]^T \; x_2 = [-1, 0, 2]^T; \; \lambda = 8. \; x_3 = [2, -1, 1]^T$.

$P = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

12 Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ Hence determine A^4

Ans:- $\lambda = -1, -1, \quad x_1 = [1, 0, -1]^T \; x_2 = [0, 1, -1]^T; \; \lambda = 2, \; x_3 = [1, 1, 1]^T$.

13. Diagonalize, if possible $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

Ans:- $\lambda = 1 \quad x_1 = [1, 1, -1]^T \quad \lambda = 2, 2, \; x_2 = [2, 1, 0]^T$ Not diagonalizable.

14 Diagonalize if possible $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Ans: $\lambda = 1, 1, 1 \quad x = [0, 3, -2]^T$. Not diagonalizable.

15 Diagonalize if possible $A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ -1 & 3 & 4\end{bmatrix}$

Ans:- $\lambda=1,1,\ x_1=[0,1,-1]^T\quad \lambda=7,\ x_2=[6,7,5]^T$ Not diagonalizable.

16 Diagonalize if possible $A=\begin{bmatrix}1 & 1 & 1\\ -1 & -3 & -3\\ 2 & 4 & 4\end{bmatrix}$

Ans:- $\lambda=0,0\quad x_1=[0,1,-1]^T\quad \lambda=2\quad x_2=[1,-2,3]^T$ Not diagonalizable.

17 Diagonalize the matrix $A=\begin{bmatrix}1 & 1 & i\\ 1 & 0 & i\\ -i & -i & 1\end{bmatrix}$ Hence determine A^7

Ans:- $\lambda=0,\ x_1=[1,0,-i]^T;\ \lambda=1+\sqrt{3},\ x_2=[1,\sqrt{3}-1,-i]^T$

$\lambda=1-\sqrt{3}\quad x_3=[1,-(\sqrt{3}+1),-i]^T$

$P=\begin{bmatrix}i & 1 & 1\\ 0 & \sqrt{3}-1 & -1-\sqrt{3}\\ -1 & -i & -i\end{bmatrix}\quad P^{-1}=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

18 Diagonalize the matrix $A=\begin{bmatrix}1 & -1 & 1\\ 1 & 0 & 0\\ -1 & 1 & -1\end{bmatrix}$ Hence determine A^4

Ans:- $\lambda=0,\ x_1=[0,1,1]^T;\ \lambda=i,\ x_2=[1,-i,-1]^T;\ \lambda=-i,\ x_3=[1,i,-1]^T$

$P=\begin{bmatrix}0 & 1 & 1\\ 1 & -i & i\\ 1 & -1 & -1\end{bmatrix}\quad P^{-1}=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

19 Diagonalize the matrix $A=\begin{bmatrix}-1 & 2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$ Hence determine A^5

Ans:- $\lambda=1,\ x_1=[1,0,-1]^T;\ \lambda=\sqrt{5},\ x_2=[\sqrt{5}-1,1,-1]^T;\ \lambda=-\sqrt{5},\ x_3=[\sqrt{5}+1,-1,1]^T$

$P=\begin{bmatrix}1 & \sqrt{5}-1 & \sqrt{5}+1\\ 0 & 1 & -1\\ -1 & -1 & 1\end{bmatrix}\quad P^{-1}=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

20 Diagonalize the matrix $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$ Hence determine A^6

Ans:- $\lambda=1,1\quad x_1=[1,1,0]^T\quad x_2=[1,0,1]^T;\ \lambda=-2,\ x_3=[-1,1,1]^T$

$P=\begin{bmatrix}1 & 1 & -1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}\quad P^{-1}=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

→ Find the matrix A whose eigen values are 1, −1, 2 and corresponding eigen vectors are $[1\ 1\ 0]^T$, $[1\ 0\ 1]^T$ and $[3\ 1\ 1]^T$.

Sol: G.T an eigen values of A are $\lambda_1 = 1$, $\lambda_2 = -1$ $\lambda_3 = 2$

$$\text{spectral matrix } D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$$\text{Modal matrix } P = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(0-1) - 1(1-0) + 3(1-0)$$

$$|P| = 1$$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P$$

$$P^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

We have $A^n = P D^n P^{-1}$

$$A = P D P^{-1}$$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -5 & -7 \\ 1 & 0 & -1 \\ 3 & -3 & -4 \end{bmatrix}$$

DIAGONALIZATION OF A MATRIX

1 Find the matrix A whose eigen values and corresponding eigen vectors are as given below.

(a) Eigen values 2, 2, 4; Eigen vectors $[-2,1,0]^T$, $[-1,0,1]^T$, $[1,0,1]^T$

Ans:- $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

2

(b) Eigen values 1, −1, 2; Eigen vectors $[1,1,0]^T$, $[1,0,1]^T$, $[3,1,1]^T$

Ans:- $A = \begin{bmatrix} 6 & -5 & -7 \\ 1 & 0 & -1 \\ 3 & -3 & -4 \end{bmatrix}$

(c) Eigen values 1, 2, 3; Eigen vectors $[1,2,1]^T$, $[2,3,4]^T$, $[1,4,9]^T$

Ans:- $A = \frac{1}{12}\begin{bmatrix} 30 & -12 & 6 \\ 2 & 4 & 14 \\ -34 & 4 & 38 \end{bmatrix}$

(d) Eigen values 0, −1, 1; Eigen vectors $[-1,1,0]^T$, $[1,0,-1]^T$, $[1,1,1]^T$

Ans:- $A = \frac{1}{3}\begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$

(e) Eigen values 0, 0, 3; Eigen vectors $[1,2,-1]^T$, $[-2,1,0]^T$, $[3,0,1]^T$

Ans:- $A = \frac{1}{8}\begin{bmatrix} 9 & 18 & 45 \\ 0 & 0 & 0 \\ 3 & 6 & 15 \end{bmatrix}$

(f) Eigen values 1, 1, 3; Eigen vectors $[1,0,-1]^T$, $[0,1,-1]^T$, $[1,1,0]^T$.

Ans:- $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

R.NO —— Q.NO

C1 − E0 —— a, d

E1 − G0 —— b, e

G1 − J0 —— c, f

Singular Value Decomposition :—

Given an $m \times n$ complex matrix A, there in general exist an $m \times m$ unitary matrix U, an $n \times n$ unitary matrix V and an $m \times n$ matrix $D = [d_{ij}]$ with $d_{ij} = 0$ for $i \neq j$ such that $A = UDV^*$ —— ①.

The representation of A as a product of U, D and V^* as given by expression ① is known as the singular value Decomposition (or factorization) of A. The elements d_{ii} in the matrix D are called the singular values of A, the columns of U are called the left singular vectors and the columns of V are called the right singular vectors.

When A is a real matrix, the matrices U and V are orthogonal matrices and D is a real matrix.

In this case, the expression ① becomes. $A = UDV^{-1} = UDV^T$ —— ②.

This expression is equivalent to the expression

$$D = U^{-1}AV = U^TAV \text{ —— ③}.$$

When U and V are known, this expression may be employed to obtain D.

Working Procedure :—

Step 1 : Given the matrix A, obtain the matrices $B = AA^T$ and $C = A^TA$.

Step 2 : obtain the eigen values and corresponding eigen vectors of B. Deduce an orthonormal system from these eigen vectors. Form the orthogonal matrix whose columns are the vectors of this orthonormal system. Denote this orthogonal matrix by U.

Step 3 :- Proceed as in step 2 for the matrix C and obtain the orthogonal matrix V.

Step 4 :- obtain the matrix D by using $D = U^TAV$.

Step 5 :- with U, V and D as determined above, write down the singular value decomposition of A as $A = UDV^T$.

Note :- In the singular value decomposition obtained by the above-mentioned working rule, the elements of D would be such that, for each i, the element d_{ii}^2 = one of the eigen values of B.

Obtain a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

Sol:- Given that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

$$B = AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$C = A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The characteristic equation of B is $|B - \lambda I| = 0$.

$$\text{i.e} \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 9-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)(-\lambda)(9-\lambda) = 0$$

$$\lambda = 0, 1, 9.$$

$\therefore$ The eigen values of B are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 9$

Let $X = [x \ y \ z]^T$ Then the matrix equation $[B - \lambda I]X = 0$.

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 9-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case(i):- An eigen vector corresponding to the eigen value $\lambda = 0$:-

For $\lambda = 0$, the system ① can be written as .

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $x = 0$, $z = 0$.

choose. $y = k_1$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ is the linearly independent eigen vector corresponding to eigen value $\lambda = 0$.

$$e_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ is the normalized eigen vector

Case(ii):- An eigen vector corresponding to the eigen value $\lambda = 1$

For $\lambda = 1$, the system ① can be written as .

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $y = 0$, $z = 0$.

Choose $x = k_2$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the linearly independent eigen vector corresponding to eigen value $\lambda = 1$.

$$e_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the normalized eigen vector .

Case (iii) :- An eigen vector corresponding to the eigen value $\lambda = 9$

For $\lambda = 9$, The system (1) can be written as .

$$\begin{bmatrix} -8 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $x = 0$, $y = 0$

choose $z = k_3$.

$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k_3 \end{bmatrix} = k_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 9$.

$e_3 = \frac{X_3}{\|X_3\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the normalized eigen vector.

We observe that e_1, e_2 and e_3 are pairwise orthogonal and therefore these form an orthonormal system .

$$U = [e_1 \ e_2 \ e_3] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the matrix $C = A^T A$, the characteristic equation is $|C - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 9-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

$$(1-\lambda)(9-\lambda)(-\lambda) = 0$$

$$\lambda = 0, 1, 9 .$$

$\therefore$ The eigen values of the matrix C are $\lambda = 0, 1, 9$.

If $X = [x \ y \ z]^T$, the matrix equation $[C - \lambda I]X = 0$

i.e $\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 9-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case (i) :- An eigen vector corresponding to the eigen value $\lambda = 0$.

For $\lambda = 0$, The system ② can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $x = 0$, $y = 0$

choose $z = k_1$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 0$.

$$e_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ is the normalized eigen vector.

Case (ii) :- An eigen vector corresponding to the eigen value $\lambda = 1$.

For $\lambda = 1$, The system ② can be written as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $y = 0$, $z = 0$

choose $x = k_2$.

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$.

$$e_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the normalized eigen vector.

Case (iii) :- An eigen vector corresponding to the eigen value $\lambda = 9$.

For $\lambda = 9$, The system ② can be written as

$$\begin{bmatrix} -8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $x = 0$, $z = 0$.

choose $y = k_3$.

$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ k_3 \\ 0 \end{bmatrix} = k_3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the linearly independent eigen vectors corresponding to the eigen value $\lambda = 1$.

$e_3 = \frac{X_3}{\|X_3\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the normalized eigen vector.

We observe that e_1, e_2, e_3 are pairwise orthogonal.

Therefore, these form an orthonormal system.

$$V = [e_1\ e_2\ e_3] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We find that $D = U^TAV = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$A = UDV^T$ represents the singular value decomposition of the given matrix A.

Obtain the singular value decomposition of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

Sol: Given that $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$.

$$B = AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

$$C = A^TA = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

For the matrix B, the characteristic equation is $|B - \lambda I| = 0$.

i.e $\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = 0$.

$(11-\lambda)^2 - 1 = 0 \implies \lambda^2 - 22\lambda + 120 = 0$

$\lambda = 12, 10$.

$\therefore$ The Eigen values of the matrix B are $\lambda_1 = 12$, $\lambda_2 = 10$.

Let $X = [x \ y]^T$. Then the matrix equation $(B-\lambda I)X = 0$

i.e $\begin{bmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ —— (1)

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Case(i):- An eigen vector corresponding to the eigen value $\lambda = 12$.

For $\lambda = 12$, The system (1) can be written as.

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this, $-x + y = 0$.

choose $y = k_1$

$x = y = k_1$

$X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 12$

$\|X_1\| = \sqrt{1+1} = \sqrt{2}$.

$e_1 = \dfrac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ is the normalized eigen vector.

Case(ii):- An eigen vector corresponding to the eigen value $\lambda = 10$

For $\lambda = 10$, The system (1) can be written as

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this, $x + y = 0$

choose $x = k_2$

$y = -x = -k_2$

$X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_2 \\ -k_2 \end{bmatrix} = k_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value. $\lambda = 10$.

$\|x_2\| = \sqrt{1+1} = \sqrt{2}.$

$e_2 = \dfrac{x_2}{\|x_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ is the linearly independent eigen vector normalized.

We observe that the eigen vectors e_1 and e_2 are orthogonal.

Therefore these form an orthonormal system,

$$U = [x_1 \;\; x_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

→ For the matrix $C = A^TA$, the characteristic equation is $|C-\lambda I| = 0$

$$\begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} = 0.$$

$$(10-\lambda)\left[(10-\lambda)(2-\lambda) - 16\right] + 2\left[0 - 2(10-\lambda)\right] = 0$$

$$(10-\lambda)(\lambda^2 - 12\lambda) = 0.$$

$$\lambda = 12,\ 10,\ 0.$$

∴ The eigen values of the matrix are 12, 10, 0.

Let $X = [x\ y\ z]^T$ Then the matrix equation $[C-\lambda I]\{X = 0$.

$$\text{i.e} \quad \begin{bmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——— (2).}$$

Case (i):- An eigen vector corresponding to the eigen value $\lambda = 12$:-

For $\lambda = 12$, The system (2) can be written as

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 0 & 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $-x+z=0$

$$-y+2z=0$$

Choose $z=K_1$

$$x=z=K_1$$

$$y=2z=2K_1$$

$X_1=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}K_1\\2K_1\\K_1\end{bmatrix}=K_1\begin{bmatrix}1\\2\\1\end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda=12$

$$\|X_1\|=\sqrt{1+4+1}=\sqrt{6}$$

$e_1=\dfrac{X_1}{\|X_1\|}=\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$ is the normalized eigen vector.

Case(ii) :- An eigen vector corresponding to the eigen value $\lambda=10$:—

For $\lambda=10$, The system (1) can be written as

$$\begin{bmatrix}0&0&2\\0&0&4\\2&4&-8\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_1\leftrightarrow R_3$

$$\begin{bmatrix}2&4&-8\\0&0&4\\0&0&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3\rightarrow 2R_3-R_2$

$$\begin{bmatrix}2&4&-8\\0&0&4\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

From this, $x+2y-4z=0$

$$z=0$$

$$x+2y=0$$

choose $y=K_2$

$$x=-2y=-2K_2$$

$X_2=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2K_2\\K_2\\0\end{bmatrix}=-K_2\begin{bmatrix}2\\-1\\0\end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda=10$

$$\|X_2\|=\sqrt{4+1+0}=\sqrt{5}$$

$e_2=\dfrac{X_2}{\|X_2\|}=\begin{bmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{bmatrix}$ is the normalized eigen vector.

case(iii): An eigen vector corresponding to the eigen value $\lambda = 0$:-

For $\lambda = 0$, The system ② can be written as

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 5R_3 - R_1$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 0 & 20 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $5x + z = 0$

$5y + 2z = 0$.

choose $x = k_3$

$z = -5x = -5k_3$

$5y = -2z = 10k_3$

$y = 2k_3$

$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_3 \\ 2k_3 \\ -5k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 0$.

$\|X_3\| = \sqrt{1+4+25} = \sqrt{30}$

$e_3 = \dfrac{X_3}{\|X_3\|} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}$ is the normalized eigen vector.

We observe that $e_1, e_2\ e_3$ are pairwise orthogonal

Therefore they form an orthonormal system.

$$V = [X_1\ X_2\ X_3] = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

• $D = U^T A V$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 2\sqrt{2} & -\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

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$$= \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

Thus for the given matrix A, the singular value decomposition is

$A = UDV^T$.

Obtain the singular value Decomposition of the matrix $A = \begin{bmatrix} 1/\sqrt{2} & -\sqrt{3/2} \\ \sqrt{2} & 0 \end{bmatrix}$

Sol:- Given that $A = \begin{bmatrix} 1/\sqrt{2} & -\sqrt{3/2} \\ \sqrt{2} & 0 \end{bmatrix}$

$$B = AA^T = \begin{bmatrix} 1/\sqrt{2} & -\sqrt{3/2} \\ \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ -\sqrt{3/2} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$C = A^TA = \begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ -\sqrt{3/2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -\sqrt{3/2} \\ \sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 5/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 3/2 \end{bmatrix}$$

The characteristic equation of B is $|B-\lambda I| = 0$ i.e $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$.

$$(2-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0$$
$$(\lambda-3)(\lambda-1) = 0$$
$$\lambda = 3, 1.$$

∴ The eigen values of the matrix B are $\lambda_1 = 3$, $\lambda_2 = 1$

Let $X = [x\ y]^T$ Then the matrix equation $[B-\lambda I]X = 0$.

i.e $\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ —— ①

Case(i):- An eigen vector corresponding to the eigen value $\lambda_1 = 3$:-

For $\lambda = 3$, The system ① can be written as

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From this, $-x+y=0$

choose $y=k_1$

$x=y=k_1$.

$X_1=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}k_1\\k_1\end{bmatrix}=k_1\begin{bmatrix}1\\1\end{bmatrix}$ is an eigen vector corresponding to the eigen value $\lambda=3$.

$$\|X_1\|=\sqrt{1+1}=\sqrt{2}.$$

$e_1=\dfrac{X_1}{\|X_1\|}=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ is a normalized eigen vector.

Case (ii):- An eigen vector corresponding to the eigen value $\lambda=1$:-

For $\lambda=1$, The system ① can be written as

$$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$R_2\to R_2-R_1$

$$\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

From this, $x+y=0$

choose $x=k_2$

$y=-x=-k_2$

$X_2=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}k_2\\-k_2\end{bmatrix}=k_2\begin{bmatrix}1\\-1\end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda=1$.

$$\|X_2\|=\sqrt{1+1}$$

$e_2=\dfrac{X_2}{\|X_2\|}=\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$ is the normalized eigen vector.

We observe that e_1 and e_2 are orthogonal and therefore they form an orthonormal system.

$$U=[e_1\ e_2]=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}.$$

$\longrightarrow$ The characteristic equation of the matrix $C = A^T A$ is $|C - \lambda I| = 0$

i.e $\begin{vmatrix} \frac{5}{2} - \lambda & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{3}{2} - \lambda \end{vmatrix} = 0$ i.e $\begin{vmatrix} 5 - 2\lambda & -\sqrt{3} \\ -\sqrt{3} & 3 - 2\lambda \end{vmatrix} = 0$ 13

$$(5 - 2\lambda)(3 - 2\lambda) - 3 = 0$$

$$4\lambda^2 - 16\lambda + 12 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 3, 1$$

$\therefore$ The eigen values of the matrix C is are $\lambda = 3, \lambda = 1$.

Let $X = [x \;\; y]^T$ Then the matrix equation $[C - \lambda I]X = 0$.

i.e $\begin{bmatrix} 5 - 2\lambda & -\sqrt{3} \\ -\sqrt{3} & 3 - 2\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ —— ②

Case (i) :- An eigen vector corresponding to the eigen value $\lambda = 3$:-

For $\lambda = 3$, Then system ② can be written as

$$\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \sqrt{3} R_1$$

$$\begin{bmatrix} -1 & -\sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this $\quad -x + \sqrt{3} y = 0$.

choose $y = K_1$

$x = -\sqrt{3} y = -\sqrt{3} K_1$

$X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\sqrt{3} K_1 \\ K_1 \end{bmatrix} = -K_1 \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 3$.

$$\|X_1\| = \sqrt{3 + 1} = 2$$

$e_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$ is the normalized eigen vector.

Case (ii) An eigen vector corresponding to the eigen value $\lambda = 1$:—

For $\lambda = 1$, The system ② can be written as .

$$\begin{bmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \longrightarrow R_2 + \frac{1}{\sqrt{3}} R_1$$

$$\begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this, $\sqrt{3}\,x - y = 0$.

choose $x = k_2$

$y = \sqrt{3}\,x = \sqrt{3}\,k_2$

$X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_2 \\ \sqrt{3}\,k_2 \end{bmatrix} = k_2\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ is the linearly independent eigen vector correspon-ding to the eigen value $\lambda = 1$.

$\|X_2\| = \sqrt{1+3} = 2$.

$e_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$ is the normalized eigen vector.

We observe that e_1 and e_2 are orthogonal and therefore they form an orthonormal system.

$$V = [e_1 \ e_2] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

We find that $D = U^T A V$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} \\ \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{\frac{3}{2}} & \frac{-1}{\sqrt{2}} \\ \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -1 \end{bmatrix}$$

With U, V and D as determined above $A = UDV^T$ gives singular value decomposition .

Sylvester's Theorem :—

This theorem is useful to find the approximate value of a matrix to a higher power and functions of matrices.

If the square matrix A has n distinct eigen values $\lambda_1, \lambda_2 \cdots \lambda_n$ and P(A) is a polynomial of the form.

$$P(A) = C_0 A^n + C_1 A^{n-1} + C_2 A^{n-2} + \cdots + C_{n-1} A + C_n I_n .$$

Where $C_0\ C_1\ C_2 \cdots C_n$ are constants then the polynomial P(A) can be expressed in the following form.

$$P(A) = \sum_{r=1}^{n} P(\lambda_r) \cdot Z(\lambda_r) = P(\lambda_1) Z(\lambda_1) + P(\lambda_2) Z(\lambda_2) + \cdots + P(\lambda_n) Z(\lambda_n)$$

$$\text{Where } Z(\lambda_r) = \frac{[f(\lambda_r)]}{f'(\lambda_r)}.$$

Here $f(\lambda) = |\lambda I - A|$

$[f(\lambda)]$ = Adjoint of the matrix $[\lambda I - A]$.

and $f'(\lambda_r) = \left(\frac{df}{d\lambda}\right)_{\lambda = \lambda_r}$.

(1) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, find A^{50}.

Sol:- Consider the polynomial $P(A) = A^{50}$.

$$\text{Now } [\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{bmatrix}$$

$$f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{vmatrix}$$

$$f(\lambda) = (\lambda-1)(\lambda-3) = \lambda^2 - 4\lambda + 3. \quad \text{—— (1)}$$

∴ Eigen values of $f(\lambda)$ are $\lambda_1 = 1$ and $\lambda_2 = 3$.

From (1) $f'(\lambda) = 2\lambda - 4$ —— (2)

$$f'(\lambda_1) = f'(1) = -2$$

$$f'(\lambda_2) = f'(3) = 6 - 4 = 2.$$

$[f(\lambda)]$ = Adjoint matrix of the matrix $[\lambda I - A]$

$$[f(\lambda)] = \begin{bmatrix} \lambda-3 & 0 \\ 0 & \lambda-1 \end{bmatrix} \quad \text{——} (3)$$

$$z(\lambda_r) = \frac{[f(\lambda_r)]}{f'(\lambda_r)} \quad r = 1, 2., \text{ we get}$$

$$z(\lambda_1) = \frac{[f(\lambda_1)]}{f'(\lambda_1)} \qquad z(\lambda_2) = \frac{[f(\lambda_2)]}{f'(\lambda_2)}$$

$$\therefore z(\lambda_1) = z(1) = \frac{[f(1)]}{f'(1)} = -\frac{1}{2}\begin{bmatrix} 1-3 & 0 \\ 0 & 1-1 \end{bmatrix}$$

$$z(\lambda_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$z(\lambda_2) = z(3) = \frac{[f(3)]}{f'(3)} = \frac{1}{2}\begin{bmatrix} 3-3 & 0 \\ 0 & 3-1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ By Sylvester's theorem, we get

$$P(A) = P(\lambda_1)\, z(\lambda_1) + P(\lambda_2)\, z(\lambda_2)$$

$$A^{50} = \lambda_1^{50}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3^{50}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 1 & 0 \\ 0 & 3^{50} \end{bmatrix}$$

<u>Theorem</u> :- The sum of the eigen values of a square matrix is equal to its trace.

<u>proof</u> :- We shall prove this theorem by considering a square matrix of order 3.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3 and λ be its eigen value

We prove that $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$.

The characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

Expand it by using R_1, we have.

$$|A-\lambda I| = (a_{11}-\lambda)\left[(a_{22}-\lambda)(a_{33}-\lambda) - a_{32}a_{23}\right] - a_{12}\left[a_{21}(a_{33}-\lambda) - a_{31}a_{23}\right]$$
$$+ a_{13}\left[a_{21}a_{32} - a_{31}(a_{22}-\lambda)\right]$$
$$= (a_{11}-\lambda)\left[a_{22}a_{33} - a_{22}\lambda - a_{33}\lambda + \lambda^2 - a_{32}a_{23}\right] - a_{12}\left[a_{21}a_{33} - a_{21}\lambda\right.$$
$$\left.- a_{31}a_{23}\right] + a_{13}\left[a_{21}a_{32} - a_{31}a_{22} + a_{31}\lambda\right]$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{22}\lambda - a_{11}a_{33}\lambda + a_{11}\lambda^2 - a_{11}a_{23}a_{32} - a_{22}a_{33}\lambda$$
$$+ a_{22}\lambda^2 + a_{33}\lambda^2 - \lambda^3 + a_{23}a_{32}\lambda - a_{12}a_{21}a_{33} + a_{12}a_{21}\lambda - a_{31}a_{23}a_{12}$$
$$+ a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22} + a_{13}a_{31}\lambda.$$
$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(a_{11}a_{22} + a_{11}a_{33} + a_{22} - a_{33} - a_{23}a_{32}$$
$$- a_{12}a_{21} - a_{13}a_{31}) + (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$
$$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}) \quad \text{--- (1)}$$

If $\lambda_1 \lambda_2 \lambda_3$ are the eigen values of A then.

$$|A-\lambda I| = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$
$$|A-\lambda I| = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) \quad \text{--- (2)}$$

$$|A-\lambda I| = \left[\lambda_1\lambda_2 - \lambda_1\lambda - \lambda_2\lambda + \lambda^2\right](\lambda_3-\lambda)$$

$$= \lambda_1\lambda_2\lambda_3 - \lambda\lambda_1\lambda_3 - \lambda\lambda_2\lambda_3 + \lambda^2\lambda_3 - \lambda\lambda_1\lambda_2 + \lambda^2\lambda_1 + \lambda^2\lambda_2 - \lambda^3$$

$$|A-\lambda I| = -\lambda^3 + \lambda^2(\lambda_1+\lambda_2+\lambda_3) - \lambda(\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1) + \lambda_1\lambda_2\lambda_3 \quad \text{—— (3)}$$

Equating the R.H.S of (1) and (3) and comparing the coefficients of λ^2, we have.

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}.$$

i.e The sum of the eigen values of A = The sum of the elements of the principal diagonal of A.

Hence The sum of the eigen values of a matrix A is equal to the trace of the matrix A.

(OR)

Another Proof :-

Let A be square matrix of order n.

The characteristic equation of A is $|A-\lambda I| = 0$

$$\text{i.e} \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Expanding this, we get

$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) - a_{12}$ (a polynomial of degree $n-2$)

$+ a_{13}$ (a polynomial of degree $n-2$) $+ \cdots = 0.$

$(-1)^n(\lambda-a_{11})(\lambda-a_{22})\cdots(\lambda-a_{nn})$ + a polynomial of degree $(n-2) = 0$.

$(-1)^n\left[\lambda^n - (a_{11}+a_{22}+a_{33}+\cdots+a_{nn})\lambda^{n-1}\right.$ + a polynomial of degree $(n-2)$ in $\lambda = 0$.

$(-1)^n\lambda^n + (-1)^{n+1}(\text{Trace } A)\lambda^{n-1}$ + a polynomial of degree $(n-2)$ in $\lambda = 0$.

If $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$ are the roots of this equation

$$a = (-1)^n \quad b = (-1)^{n+1} \cdot \text{Trace } A$$

$$\text{Sum of the roots} = \frac{-b}{a}$$

$$= \frac{-(-1)^{n+1}\text{Trace } A}{(-1)^n} = \text{Trace } A$$

$$\therefore \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n) = \text{Trace } A$$

Hence The sum of eigen values of a matrix A is equal to the trace of the matrix A.

<u>Theorem</u> : The Product of the eigen values of a matrix is equal to its determinant.

Proof :- Let $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$ be the eigen values of square matrix A of order n.

We prove that $\lambda_1 \lambda_2 \lambda_3 \dots \lambda_n = \det A$

The characteristic polynomial of A is

$$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \dots (\lambda - \lambda_n) \text{ --- (i)}$$

Taking $\lambda = 0$ in (i), we have

$$|A| = (-1)^n (0 - \lambda_1)(0 - \lambda_2) \dots (0 - \lambda_n)$$

$$|A| = (-1)^n (-1)^n \lambda_1 \lambda_2 \lambda_3 \dots \lambda_n$$

$$|A| = (-1)^{2n} \lambda_1 \lambda_2 \lambda_3 \dots \lambda_n$$

$$|A| = \lambda_1 \lambda_2 \lambda_3 \dots \lambda_n \quad [\because (-1)^{2n} = 1]$$

i.e The det. of A = The product of the eigen values of A.

Hence the product of the eigen values of A is equal to its determinant.

Note :- (i) If one of the eigen values of a matrix A is zero then det A = 0 i.e A is singular matrix and vice versa.

(ii) If all the eigen values of a matrix A are non zero then det A ≠ 0 i.e A is non singular matrix and vice versa.

* - Theorem 3 :- If λ is an eigen value of A corresponding to the eigen vector X then λ^n is an eigen value of A^n corresponding to the eigen vector X.

Proof :- Given that λ is an eigen value of a matrix A and X be its corresponding eigen vector.

We P/T λ^n is an eigen value of A^n corresponding to the eigen vector X.

We prove this by using mathematical induction.

By definition, λ is an eigen value of A if there exists a non-zero vector such that $AX = \lambda X$ —— ①

The Result is true for n = 1.

Pre multiplying eqn ① bothsides with A, we get

$$A(AX) = A(\lambda X)$$

$$A^2X = \lambda(AX)$$

$$A^2X = \lambda(\lambda X)$$

$$A^2X = \lambda^2 X \quad \text{——} ②$$

Hence λ^2 is an eigen value of A^2 with "X" itself as the corresponding eigen vector.

Thus the theorem is true for n = 2.

Let the result is true for n = k.

$$A^kX = \lambda^k X \quad \text{——} ③$$

Pre multiplying eqn ③ bothsides with A, we get.

$$A(A^kX) = A(\lambda^k X)$$

$$A^{k+1}X = \lambda^k(AX)$$

$$A^{k+1}x = \lambda^{k+1}x.$$

Which implies that λ^{k+1} is an eigen value of A^{k+1} with x itself as the corrosponding eigen vector.

Hence by the principle of mathematical induction, the theorem is true for all positive integer n.

Hence λ is an eigen value of A corrosponding to the eigen vector x then λ^n is an eigen value of A^n corrosponding to the eigen vector x

Theorem :- A square matrix A and its transpose A^T have the same eigen values.

Proof :- Let λ be an eigen value of the matrix A.

We prove that λ is an eigen value of the matrix A^T.

We know that For any square matrix B, $|B| = |B^T|$.

$$(A-\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I.$$

We have $|A-\lambda I| = |(A-\lambda I)^T|$

$$|A-\lambda I| = |A^T - \lambda I^T|$$

$$|A-\lambda I| = |A^T - \lambda I|$$

$\therefore$ $|A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$.

i.e λ is an eigen value of A if and only if λ is an eigen value of A^T

Hence the Eigen values of A and A^T are same.

1) Verify that sum of the eigen values of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ is equal to its trace. and also verify that product of the eigen values of the matrix A is equal to its determinant.

Sol.- The characteristic equation of the matrix A is $|A-\lambda I| = 0$ i.e.

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

Where S_1 = Sum of the principal diagonal elements of $A = 1+2+3 = 6$.

S_2 = Sum of the minors of principal diagonal elements of A.

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (6-2) + (3+2) + (2-0)$$

$$S_2 = 11$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1(6-1) - 0 - 1(2-4)$$

$$S_3 = 6.$$

Hence the characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$$\lambda = 1, 2, 3.$$

(i) Sum of the eigen values of A is $1+2+3 = 6$.

Trace of A is $1+2+3 = 6$.

$\therefore$ Sum of the eigen values = Trace of A.

(ii) Product of the eigen values of A is $1.2.3 = 6$.

$\det(A) = 6$.

$\therefore$ Product of eigen values = $\det(A)$.

2) Verify that an eigen values of A and A^T are same where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Sol:- The characteristic equation of A is $|A-\lambda I| = 0$

$$\text{i.e.} \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

Where S_1 = sum of the principal diagonal elements of $A = 1+2+3 = 6$.

S_2 = Sum of the minors of principal diagonal elements of A. 4

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (6-2) + (3+2) + (2-0)$$

$$S_2 = 11$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \end{vmatrix} = 1(6-1) - 0 - 1(2-4)$$

$$S_3 = 6$$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$$\lambda = 1, 2, 3.$$

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

The characteristic equation of A^T is $|A^T - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\left[(2-\lambda)(3-\lambda)-2\right] - 1\left[2-2(2-\lambda)\right] = 0.$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 4) - (2\lambda - 2) = 0.$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0.$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

$$\lambda = 1, 2, 3.$$

We observe that eigen values of A and A^T are same.

3) Find the eigen values of the matrix A^2, where $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{vmatrix} = 0.$

$$(2-\lambda)(-1-\lambda)-4=0.$$
$$(1+\lambda)(\lambda-2)-4=0$$
$$\lambda^2-\lambda-6=0.$$
$$\lambda=3,-2.$$

We know that λ is an eigen value of A corresponding to the eigen vector X then λ^n is an eigen value of A^n corresponding to the eigen vector X.

$\therefore$ The eigen values of A^2 are 3^2, $(-2)^2$ i.e 9, 4.

Theorem:- If $\lambda_1\ \lambda_2\ \lambda_3 \ldots \lambda_n$ are the eigen values of a matrix A then $K\lambda_1, K\lambda_2, \ldots K\lambda_n$ are the eigen values of the matrix KA where K is a non zero scalar.

Proof:- Given that $\lambda_1\ \lambda_2\ \lambda_3 \ldots \lambda_n$ are the eigen values of matrix A

We prove that $K\lambda_1, K\lambda_2, \ldots K\lambda_n$ are the eigen values of matrix KA.

Let A be a square matrix of order n.

Then $|KA-\lambda KI| = |KA-K\lambda I|$

$$=|K(A-\lambda I)| \qquad (\because |KA| = K^n|A|)$$

$$|KA-\lambda KI| = K^n|A-\lambda I|$$

Since $K\neq 0$, Therefore $|KA-\lambda KI|=0$ iff $|A-\lambda I|=0$.

i.e $K\lambda$ is an eigen value of KA iff λ is an eigen value of A.

Thus $K\lambda_1, K\lambda_2, \ldots K\lambda_n$ are the eigen values of KA if $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the eigen values of A.

If $A=\begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}$ then find the eigen values of $2A$.

Sol:- The characteristic equation of A is $|A-\lambda I|=0$

$$\text{i.e}\begin{vmatrix}5-\lambda & 4\\1 & 2-\lambda\end{vmatrix}=0$$

$$(5-\lambda)(2-\lambda)-4=0.$$

$$\lambda^2-7\lambda+6=0$$

$$\lambda=1,6.$$

We know that If λ is an eigen value of A then $k\lambda$ is an eigen value KA.

$\therefore$ The eigen values of $2A$ is 2λ i.e $2, 12$.

Theorem:- If λ is an eigen value of the matrix A then $\lambda+K$ is an eigen value of the matrix $A+KI$.

Proof:- Given that λ is an eigen value of the matrix A.

We prove that $\lambda+K$ is an eigen value of the matrix $A+KI$.

Let λ be an eigen value of A and X be the corresponding an eigen vector.

Then by the definition, $AX=\lambda X$ —— ①.

Now $(A+KI)X = AX+KIX$

$= \lambda X+KX$

$(A+KI)X=(\lambda+K)X$ —— ② ($\because$ From ①)

$\left[\because AX=\lambda X\right.$

$AX+KX=\lambda X+KX$

$\left[AX+KIX=\lambda X+KX\right.$

$\left[(A+KI)X=(\lambda+K)X\right.$

$\therefore$ By def, From ②, This show that the scalar $\lambda+K$ is an eigen value of the matrix $A+KI$ and X is an corresponding eigen vector.

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then find the eigen values of $A+30I$.

sol:- The characteristic equation of A is $|A-\lambda I| = 0$ i.e $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(2-\lambda) - 4 = 0.$$

$$\lambda^2 - 7\lambda + 6 = 0.$$

$$\lambda = 1, 6.$$

$\lambda = 1, 6$ are the eigen values of A.

We know that If λ is an eigen value of A then $\lambda+K$ is an eigen value of $A+KI$.

$\therefore$ The eigen values of the matrix $A+30I$ is $\lambda+30$

i.e 31, 36.

Theorem:- If $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the eigen values of A then $\lambda_1 - K, \lambda_2 - K, \lambda_3 - K, \ldots \lambda_n - K$ are the eigen values of the matrix $(A-KI)$ where K is a non zero scalar.

Proof:- Given that $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the eigen values of A.

We prove that $\lambda_1 - K, \lambda_2 - K \ldots \lambda_n - K$ are the egien values of $A-KI$.

The characteristic polynomial of A is.

$$|A-\lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda) \quad \text{--- (1)}.$$

Thus the characteristic polynomial of $A-KI$ is.

$$|A-KI-\lambda I| = |A-(K+\lambda)I|$$

$$= (\lambda_1 - (\lambda+K))(\lambda_2 - (\lambda+K)) \ldots (\lambda_n - (\lambda+K))$$

$$= ((\lambda_1 - K) - \lambda)((\lambda_2 - K) - \lambda) \ldots ((\lambda_n - K) - \lambda)$$

This show that the eigen values of $A-KI$ are $\lambda_1 - K, \lambda_2 - K \ldots \lambda_n - K$.

(OR)

Given that $\lambda_1 \lambda_2 \lambda_3 \dots \lambda_n$ are the eigen values of the matrix A.

We prove that $\lambda_1 - K, \lambda_2 - K \dots \lambda_n - K$ are the eigen values of $A - KI$.

Let λ be an eigen value of A and X be the corrosponding eigen vector.

Then by the definition, $AX = \lambda X$ —— (1).

$$\begin{aligned} \text{Now } (A-KI)X &= AX - KIX \\ &= \lambda X + KX. \quad (\because \text{From (1)}) \\ (A-KI)X &= (\lambda - K)X. \quad \text{—— (2)} \end{aligned}$$

$$\left[\begin{aligned} &\because AX = \lambda X \\ &AX + KX = \lambda X + KX \\ &AX + KIX = \lambda X + KX \\ &(A+KI)X = (\lambda - K)X \end{aligned}\right.$$

$\because$ By def, From (2). This show that the scalar $\lambda - K$ is an eigen value of the matrix $A - KI$ and X is a corrosponding eigen vector.

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then find the eigen values of $A - 44I$ and $A + 2I$.

sol: The characteristic equation of A is $|A - \lambda I| = 0$ i.e $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0.$$

$$\lambda = 1, 6.$$

$\lambda = 1, 6$ are the eigen values of A.

We know that If λ is an eigen value of A then $\lambda - K$ is an eigen value of $A - KI$.

$\therefore$ The eigen values of the matrix $A - 44I$ is $\lambda - 44$

i.e $-43, -38$

$\therefore$ The eigen values of the matrix $A + 2I$ is $\lambda + 2$

i.e $3, 8$.

Theorem :- If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the eigen values of A then $(\lambda_1-\lambda)^2\ (\lambda_2-\lambda)^2 \ldots (\lambda_n-\lambda)^2$ are the eigen values of $(A-\lambda I)^2$.

Proof :- Given that $\lambda_1\ \lambda_2\ \lambda_3 \ldots \lambda_n$ are the eigen values of A.

We prove that $(\lambda_1-\lambda)^2, (\lambda_2-\lambda)^2, \ldots (\lambda_n-\lambda)^2$ are the eigen values of $(A-\lambda I)^2$

First we prove that $\lambda_1-\lambda, \lambda_2-\lambda \ldots \lambda_n-\lambda$ are eigen values of $A-\lambda I$.

$\therefore$ The characteristic polynomial of A is

$$|A-KI| = (\lambda_1-K)(\lambda_2-K)\ldots(\lambda_n-K) \quad \text{--- (1)}.$$

Where K is a scalar.

The characteristic polynomial of $(A-\lambda I)$ is

$$|A-\lambda I-KI| = |A-(\lambda+K)I|$$
$$= [\lambda_1-(\lambda+K)][\lambda_2-(\lambda+K)]\ldots[\lambda_n-(\lambda+K)]$$
$$= [(\lambda_1-\lambda)-K][(\lambda_2-\lambda)-K]\ldots[(\lambda_n-\lambda)-K]$$

This shows that the eigen values of $A-\lambda I$ are $\lambda_1-\lambda, \lambda_2-\lambda \ldots \lambda_n-\lambda$.

Since by the known theorem, If the eigen values of A are $\lambda_1\ \lambda_2 \ldots \lambda_n$ then the eigen values of A^n are $\lambda_1^n, \lambda_2^n, \ldots \lambda_n^n$.

Thus the eigen values of $(A-\lambda I)^2$ are $(\lambda_1-\lambda)^2\ (\lambda_2-\lambda)^2 \ldots (\lambda_n-\lambda)^2$

* Theorem :- If λ is an eigen value of a non singular matrix A corresponding to the eigen vector x then λ^{-1} is an eigen value of A^{-1} and corresponding eigen vector x itself. (OR) The eigen values of A^{-1} are the reciprocals to the eigen values of A.

Proof :- Given that A is a non singular matrix i.e $\det A \neq 0$.

We know that the product of the eigen values is equals to $\det A$.

It follows that none of the eigen values of A is zero.

If λ is an eigen value of the non singular matrix A and X is the corrosponding eigen vector then 75

$$AX = \lambda X \quad \text{——— (1)}$$

Pre multiplying (1) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$IX = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X$$

$$\frac{1}{\lambda}X = A^{-1}X$$

$$A^{-1}X = \lambda^{-1}X$$

Hence by the definition of the eigen vector.

It follows that λ^{-1} is an eigen value of A^{-1} and X is the corrospon-ding eigen vector.

Find the eigen values of the matrix A^{-1} where $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$.

Sol:- The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-1-\lambda) - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$\lambda = 3, -2.$$

We know that λ is an eigen value of A corrosponding to the eigen vector X.

then λ^{-1} is an eigen value of A^{-1} corrosponding to the eigen vector X.

$\therefore$ The eigen values of A^{-1} are $3^{-1}, -2^{-1}$ i.e $\frac{1}{3}, \frac{-1}{2}$.

Theorem :- If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value.

Proof :- Let A be an orthogonal matrix

λ is an eigen value of A.

We prove that $\frac{1}{\lambda}$ is an eigen value of A.

Since by the known theorem, If λ is an eigen value of a non singular matrix A Then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Since A is an orthogonal matrix.

$$A^T A = AA^T = I$$

$$\therefore A^{-1} = A^T$$

$\therefore \lambda$ is an eigen value of A^T

Since by the known theorem, The square matrix A and its transpose A^T have the same eigen values.

Since determinants $|A-\lambda I|$ and $|A^T-\lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

* Theorem :- If λ is an eigen value of a non singular matrix A then $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A.

Proof :- Given that λ is an eigen value of a non singular matrix.

Therefore $\lambda \neq 0$.

λ is an eigen value of A if there exists a non zero vector X such that $AX = \lambda X$ —— (1)

Pre multiply eqn (1) by Adj A

$$(\text{Adj}A) AX = (\text{Adj}A) \lambda X$$

$$[(\text{Adj}A) A] X = \lambda (\text{Adj}A) X$$

$$|A| I X = \lambda (\text{Adj}A) X$$

$$|A| X = \lambda (\text{Adj}A) X$$

$$\frac{|A|}{\lambda} X = (\text{Adj}A) X$$

$$(\text{Adj}A) X = \frac{|A|}{\lambda} X.$$

$\therefore$ By def. It is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix AdjA

1) If eigen values of the matrix A are 2, 3 and 4. then find the eigen values of AdjA.

Sol:- Given that eigen values of A are 2, 3 and 4.

We know that If λ is an eigen value of A then $\frac{|A|}{\lambda}$ Is an eigen value of AdjA.

$$|A| = 2.3.4 = 24$$

$\therefore$ An eigen values of AdjA are $\frac{|A|}{\lambda} = \frac{24}{2} = 12, \frac{24}{3} = 8, \frac{24}{4} = 6.$

Theorem:- If A and P be square matrices of order n such that P is non singular. Then A and $P^{-1}AP$ have the same eigen values.

Proof:- Given that A and P be square matrices of order n.

Let $C = P^{-1}AP$

$$C - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}(AP - \lambda IP)$$

$$C - \lambda I = P^{-1}(A - \lambda I)P$$

$$|C - \lambda I| = |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}|\,|A - \lambda I|\,|P|$$

$$= |P^{-1}|\,|P|\,|A - \lambda I| \qquad [\because |P^{-1}|\,|P| = |P^{-1}P| = |I| = 1]$$

$$= |P^{-1}P|\,|A - \lambda I|$$

$$= |I|\,|A - \lambda I|$$

$$\therefore |C - \lambda I| = |A - \lambda I|$$

Thus the characteristic polynomials of C and A are same.

Hence the eigen values of $P^{-1}AP$ and A are same.

Corollary :- If A and B are square matrices of order n and A is invertible then $A^{-1}B$ and BA^{-1} have same eigen values.

Proof :- Given that A and B are square matrices of order n.

A is invertible $\Longrightarrow$ A^{-1} is exists.

We proove that $A^{-1}B$ and BA^{-1} have same eigen values.

We know that If A and P are square matrices of order n such that P is non singular then A and $P^{-1}AP$ have same eigen values.

Taking $A = BA^{-1}$ and $P = A$, we have.

BA^{-1} and $A^{-1}(BA^{-1})A$ have same eigen values.

BA^{-1} and $(A^{-1}B)(A^{-1}A)$ have same eigen values.

BA^{-1} and $(A^{-1}B)I$ have same eigen values.

BA^{-1} and $A^{-1}B$ have same eigen values.

Corollary :- If A and B are non singular matrices of the same order then AB and BA have the same eigen values.

Proof :- Given that A and B are non singular matrices of same order.

A is invertible $\Longrightarrow$ A^{-1} exists.

B is invertible $\Longrightarrow$ B^{-1} exists.

We have to P.T AB and BA have same eigen values.

We know that If A and P are square matrices of order n such that P is non singular then A and $P^{-1}AP$ have same eigen values.

Taking $A = BA$ and $P = A^{-1}$, we have.

BA and $(A^{-1})^{-1}(BA)A^{-1}$ have the same eigen values

BA and $A(BA)A^{-1}$ have the same eigen values.

BA and $(AB)(AA^{-1})$ have the same eigen values.

BA and $(AB)I$ have the same eigen values.

$\therefore$ BA and AB have same eigen values

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$ then verify that AB and BA have the same eigen values. 9

Sol:- Given that $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 16 \\ 4 & 2 \end{bmatrix}$$

The characteristic equation of AB is $|AB - \lambda I| = 0$

i.e $\begin{vmatrix} 14-\lambda & 16 \\ 4 & 2-\lambda \end{vmatrix} = 0$

$(14-\lambda)(2-\lambda) - 64 = 0$

$\lambda^2 - 16\lambda - 36 = 0.$

$\lambda = 18, -2$

$$BA = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 16 \\ 4 & 2 \end{bmatrix}$$

The characteristic equation of BA is $|BA - \lambda I| = 0$

i.e $\begin{vmatrix} 14-\lambda & 16 \\ 4 & 2-\lambda \end{vmatrix} = 0.$

$\lambda^2 - 16\lambda - 36 = 0$

$\lambda = 18, -2$

We observe that the eigen values of AB and BA are same.

Theorem:- The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof:- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & a_{nn} \end{bmatrix}_{n \times n}$ be a triangular matrix of order n.

The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ - & - & - & - & - \\ 0 & 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.$

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)=0$$

$$\lambda = a_{11}, a_{22}, \cdots a_{nn}$$

Hence the eigen values of A are $a_{11}, a_{22}, a_{33} \cdots a_{nn}$

Which are just the diagonal elements of A.

<u>Note</u>:- Similarly we can show that the eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Eg: Find the eigen values of the matrix $A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 44 & 4 \\ 0 & 0 & 30 \end{bmatrix}$

sol:- Given that $A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 44 & 4 \\ 0 & 0 & 30 \end{bmatrix}$

The given matrix A is upper triangular matrix.

$\therefore$ The eigen values are the diagonal elements of A.

$\therefore$ The eigen values of A are 2, 44 and 30.

<u>Theorem</u>:- The eigen values of a real symmetric matrix are always real or real numbers.

<u>Proof</u>:- Let A be real symmetric matrix $\Rightarrow A^T = A$.

Let λ be an eigen value of a real symmetric matrix A and let X be the corresponding eigen vector.

Then $AX = \lambda X$ —— (1)

Take the conjugate $\bar{A}\bar{X} = \bar{\lambda}\bar{X}$

Take the transpose $(\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T$

$$\bar{X}^T\bar{A}^T = \bar{\lambda}\bar{X}^T$$

$$\bar{X}^T A^T = \bar{\lambda}\bar{X}^T \quad \text{since } \bar{A} = A$$

$$\bar{X}^T A = \bar{\lambda}\bar{X}^T \quad \text{since } A^T = A$$

Post multiply by X, we have.

$$\bar{X}^T A X = \bar{\lambda}\bar{X}^T X \text{ —— (2)}$$

Pre multiply ① by $\bar{x}^T$, we get

$$\bar{x}^T A x = \bar{x}^T \lambda x \quad \text{——③}$$

② — ③ gives

$$(\bar{\lambda} - \lambda)\, \bar{x}^T x = 0$$

$$\bar{\lambda} - \lambda = 0$$

$$\lambda = \bar{\lambda}$$

$$\Rightarrow \lambda \text{ is real}$$

[Since x is non zero vector
$\bar{x}^T$ is non zero vector
$x\bar{x}^T \neq 0$

Verify that the eigen values of real symmetric matrix $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ are real.

Sol: The characteristic equation of A is $|A - \lambda I| = 0$ i.e $\begin{vmatrix} 3-\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ -2 & 0 & 0-\lambda \end{vmatrix} = 0$

$$\text{i.e } (3-\lambda)\left[(2-\lambda)(-\lambda) - 0\right] - 2\left[0 + 2(2-\lambda)\right] = 0.$$

$$-\lambda^3 + 5\lambda^2 - 2\lambda - 8 = 0.$$

$$\lambda = -1, 2, 4.$$

We observe that the eigen values of real symmetric matrix are real.

<u>Theorem</u> :— For a real symmetric matrix, The eigen vectors corresponding to two distinct eigen values are orthogonal.

<u>Proof</u> :- Let A be a real symmetric matrix.

Let λ_1 λ_2 be eigen values of a real symmetric matrix A.

Let x_1 x_2 be the corresponding eigen vectors.

We have to prove that x_1 is orthogonal to x_2 i.e $x_1^T x_2 = 0$.

Since x_1, x_2 are eigen vectors of A corresponding to the eigen values λ_1 and λ_2

$$\text{We have } \quad Ax_1 = \lambda x_1 \quad \text{——①}$$

$$Ax_2 = \lambda x_2 \quad \text{——②}.$$

Pre multiply ① by x_2^T, we get

$$x_2^T A x_1 = x_2^T \lambda_1 x_1$$

$$x_2^T A x_1 = \lambda_1 x_2^T x_1.$$

Taking transpose, we get

$$(x_2^T A x_1)^T = (\lambda_1 x_2^T x_1)^T$$

$$x_1^T A^T (x_2^T)^T = \lambda_1 x_1^T (x_2^T)^T$$

$$x_1^T A x_2 = \lambda_1 x_1^T x_2 \quad \text{—— ③}$$

Pre multiply ② by x_1^T, we get $x_1^T A x_2 = \lambda_2 x_1^T x_2$ —— ④

③ − ④, we get

$$(\lambda_1 - \lambda_2) x_1^T x_2 = 0$$

$$\Rightarrow x_1^T x_2 = 0 \qquad \text{since } \lambda_1 \neq \lambda_2$$

$\therefore$ x_1 is orthogonal to x_2.

Eg: If $[1\ 0\ -1]^T$ $[-1\ 2\ -1]^T$ are eigen vectors corresponding to two distinct eigen values of real symmetric matrix A then find the third eigen vector.

sol: Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

Let $x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the eigen vector orthogonal to x_1 and x_2.

$x_1\ x_3$ are orthogonal $\Rightarrow a + 0.b - c = 0$ —— ①.

$x_2\ x_3$ are orthogonal $\Rightarrow -a + 2b - c = 0$ —— ②.

solving ① and ②, we get

$$\begin{matrix} 0 & -1 & 1 & 0 \\ 2 & -1 & -1 & 2 \end{matrix}$$

$$\frac{a}{2} = \frac{b}{2} = \frac{c}{2} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1}.$$

$\therefore$ $x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the required third eigen vector.

Theorem :- The two eigen vectors corresponding to the two different eigen values are linearly independent.

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Proof :- Let A be a square matrix.

Let x_1 and x_2 be the two eigen vectors of A corresponding to two distinct eigen values λ_1 and λ_2 Then.

$$AX_1 = \lambda_1 X_1 \quad \text{and} \quad AX_2 = \lambda_2 X_2 \quad \text{——} (1)$$

We prove that the eigen vectors x_1 and x_2 are L.I.

Let us assume that the eigen vectors x_1 and x_2 are L.D

By def. Then for two scalars k_1 and k_2 are not both zeros.

such that $k_1 x_1 + k_2 x_2 = 0$ —— (2)

Multiply both sides of (2) by A, we get

$$A(k_1 x_1 + k_2 x_2) = A(0) = 0.$$

$$k_1 (AX_1) + k_2 (AX_2) = 0.$$

$$k_1 (\lambda_1 x_1) + k_2 (\lambda_2 x_2) \quad \text{——} (3) \qquad (\because \text{from } (1))$$

(3) $- \lambda_2$ (2), gives

$$k_1 (\lambda_1 - \lambda_2) x_1 = 0.$$

$$k_1 = 0 \qquad [\because \lambda_1 \neq \lambda_2 \text{ and } x_1 \neq 0]$$

$$\Rightarrow k_2 = 0.$$

This is contradiction to our assumption that k_1, k_2 are not zeros.

Hence our assumption x_1 and x_2 are linearly dependent is wrong

$\therefore$ x_1 and x_2 are Linearly Independent.

Theorem :— If λ is an eigen value of A then the eigen value of $B = a_0 A^2 + a_1 A + a_2 I$ is $a_0 \lambda^2 + a_1 \lambda + a_2$

Proof :- If x be an eigen vector corresponding to the eigen value λ

then $AX = \lambda X$ —— (1).

Pre multiply by A on bothsides.

$$A(AX) = A(\lambda X)$$

$$A^2 X = \lambda (AX)$$

$$A^2 X = \lambda^2 X \qquad (\because \text{from } (1))$$

By the def. This shows that λ^2 is an eigen value of A^2

We have $B = a_0 A^2 + a_1 A + a_2 I$

$$BX = (a_0 A^2 + a_1 A + a_2 I) X$$

$$= a_0 A^2 X + a_1 AX + a_2 XI.$$

$$= a_0 \lambda^2 X + a_1 \lambda X + a_2 X$$

$$BX = (a_0 \lambda^2 + a_1 \lambda + a_2) X.$$

$\therefore$ By def. This show that $a_0 \lambda^2 + a_1 \lambda + a_2$ is an eigen value of B. and the corresponding eigen vector of B is X.

Note :- If λ is an eigen value of A and $f(A)$ is any polynomial in A than the eigen value of $f(A)$ is $f(\lambda)$.

Eg:- For the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ Find the eigen values $3A^3 + 5A^2 - 6A + 2I$.

sol: The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(3-\lambda)(-2-\lambda) = 0$$

$$\lambda = 1, 3, -2.$$

We know that if λ is an eigen value of A and $f(A)$ is a polynomial in A then the eigen value of $f(A)$ is $f(\lambda)$. 12

Let $f(A) = 3A^3 + 5A^2 - 6A + 2I$.

Eigen values of $f(A)$ are $f(1)$, $f(3)$ and $f(-2)$.

$$f(1) = 3(1)^3 + 5(1)^2 - 6(1) + 2(1) = 4$$

[$\because$ The eigen values of I are 1, 1, 1]

$$f(3) = 3.3^3 + 5.3^2 - 6.3 + 2.I = 110.$$

$$f(-2) = 3(-2)^3 + 5(-2)^2 - 6(-2) + 2.1 = 10.$$

$\therefore$ Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are 4, 110, 10.

Theorem :- zero is an eigen value of a matrix iff it is singular

Proof :- Let $\lambda = 0$ is an eigen value of the matrix A

The characteristic equation of A is $|A - \lambda I| = 0$ —— ①.

$\lambda = 0$ is satisfies this equation

$$|A - 0.I| = 0$$

$$|A| = 0$$

$$\Longrightarrow A \text{ is singular}$$

Converse :- A is singular

$$\Longrightarrow |A| = 0.$$

$\lambda = 0$ satisfies the equation ①.

$\lambda = 0$ is an eigen value of A.

Theorem :- If x is an eigen vector of a square matrix A, then x can be not be corresponds to more than one eigen value of A.

Proof :- If possible x corresponds to two eigen values λ_1 and λ_2 of A.

then we have $AX = \lambda_1 X$ —— ① and $AX = \lambda_2 X$ —— ②.

$$\lambda_1 X = \lambda_2 X$$

$$(\lambda_1 - \lambda_2) X = 0 \qquad \left[\because X \neq 0 \right.$$

$$\lambda_1 - \lambda_2 = 0 \qquad \text{Eigen vector is must be non}$$

$$\lambda_1 = \lambda_2 \qquad \left.\text{zero vector}\right]$$

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Theorem :- λ is a characteristic root of a square matrix A iff there exists a non zero vector X such that $AX = \lambda X$.

Proof :- Let X be a characteristic root of A.

$$\therefore |A - \lambda I| = 0.$$

$$\Longrightarrow A - \lambda I \text{ is a singular matrix.}$$

$\therefore$ The homogeneous system of equations $(A-\lambda I)X = 0$ posseses non zero solution

i.e There exists a non zero vector X such that $(A-\lambda I)X = 0$.

$$AX - \lambda I X = 0$$

$$AX = \lambda X$$

Converse :- $AX = \lambda X$

$$(A - \lambda I) X = 0.$$

where X is a non zero vector.

$\therefore$ The system of homogeneous equations $(A-\lambda I)X = 0$ has a non zero solution.

Hence the co efficient matrix $A - \lambda I$ is singular

$$\text{i.e } |A - \lambda I| = 0.$$

This shows that λ is an eigen value of A

If 2 is an eigen value of the matrix $A=\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ find the other two eigen values. 13

Sol: Let $\lambda_1\ \lambda_2\ \lambda_3$ be the eigen values of the matrix A.

$$\lambda_1 = 2$$

Sum of the eigen values of A = sum of principal diagonal elements of A.

$$2+\lambda_2+\lambda_3 = 2+1-1$$

$$\lambda_2+\lambda_3 = 0 \quad \text{--- (1)}$$

Product of the eigen values of A = Determinant of A.

$$2\lambda_2\lambda_3 = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$$2\lambda_2\lambda_3 = -8$$

$$\lambda_2\lambda_3 = -4 \quad \text{--- (2)}.$$

Solving (1) and (2), we get

$$\lambda_2 = 2 \quad \lambda_3 = -2.$$

Hence the other two eigen values are 2, -2.

If 2, 3 are the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$ find the value of a.

Sol: Let $\lambda_1\ \lambda_2\ \lambda_3$ be the eigen values of the matrix A.

$$\lambda_2 = 2 \quad \lambda_3 = 3$$

Sum of the eigen values of A = sum of principal diagonal elements of A

$$2+3+\lambda_3 = 2+2+2$$

$$\lambda_3 = 1.$$

Product of the eigen values of A = Determinant of A

$$\lambda_1\lambda_2\lambda_3 = |A|$$

$$2.3.1 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix} \Rightarrow 6 = 8-2a.$$

$$a = 1.$$

Form the matrix whose eigen values are $\alpha-5, \beta-5, \gamma-5$ where α, β, γ are the eigen values of $A = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$

Sol: If λ_1, λ_2 and λ_3 are eigen values of the matrix A then λ_1-k, λ_2-k and λ_3-k are eigen values $A-kI$.

Required matrix $= A-5I = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{bmatrix}$$

Two eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and are $\frac{1}{5}$ times to the third. Find the eigen values.

Sol:- Let $\lambda_1\ \lambda_2\ \lambda_3$ be the eigen values of the matrix A.

$$\lambda_1 = \lambda_2$$
$$\lambda_1 = \frac{\lambda_3}{5}$$
$$\lambda_2 = \frac{\lambda_3}{5}.$$

Sum of the eigen values of A = Sum of principal diagonal elements of A

$$\lambda_1 + \lambda_2 + \lambda_3 = 2+3+2$$
$$\frac{1}{5}\lambda_3 + \frac{1}{5}\lambda_3 + \lambda_3 = 7$$
$$\frac{7}{5}\lambda_3 = 7$$
$$\lambda_3 = 5$$
$$\lambda_1 = \lambda_2 = 1.$$

Hence the eigen values of A are 1, 1, 5

Theorem :- The eigen values of a real symmetric matrix are real.

Proof :- Let A be a real symmetric matrix so that $A^T = A$ 143

Now $\bar{A} = A$ since A is real.

$A = \bar{A}$ and $A = A^T \Rightarrow \bar{A} = A^T$

$\Rightarrow (\bar{A})^T = (A^T)^T = A$

$\Rightarrow A^\theta = A$.

$\Rightarrow$ A is Hermitian matrix.

The eigen values of a Hermitian matrix are real.

Hence the eigen values of a real symmetric matrix A are real.

Theorem :- The eigen values of a real skew symmetric matrix are all purely imaginary or zero.

Proof :- Let A be a skew symmetric matrix so that $A^T = -A$.

A is real $\Rightarrow \bar{A} = A$

$\Rightarrow (\bar{A})^T = A^T$

$\Rightarrow A^\theta = -A$

$\Rightarrow$ A is skew Hermitian matrix.

We know that the eigen values of a skew Hermitian matrix are purely imaginary or zero.

$\therefore$ It follows that the eigen values of skew symmetric matrix A are purely imaginary or zero.

Theorem :- The eigen values of an orthogonal matrix are of unit modulus.

Proof :- Let A be the orthogonal matrix so that $AA^T = I = A^TA$.

Let λ be the eigen value, x be the corresponding eigen vector of A.

So that $AX = \lambda X$ ——— (1).

$$(AX)^T = (\lambda X)^T$$

$$X^T A^T = \lambda X^T \quad \text{——— (2)}$$

Multiplying (1) and (2), we get—

$$(X^T A^T)(AX) = (\lambda X^T)(\lambda X)$$

$$X^T (A^T A) X = \lambda^2 (X^T X)$$

$$X^T I X = \lambda^2 (X^T X)$$

$$X^T X = \lambda^2 (X^T X)$$

$$(1 - \lambda^2)(X^T X) = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$|\lambda| = 1$$

$\Rightarrow$ unit modulus.

The eigen values of an ~~unitary~~ orthogonal matrix are of unit modulus.

Theorem :- The Eigen values of a hermitian matrix are real.

Proof :- Let A be a hermitian matrix i.e $A^\theta = A$ and λ be the eigen value of A.

We prove that λ is real.

If λ is an eigen value of A and X is the corrosponding eigen vector then $AX = \lambda X$ —— ①.

Pre multiply both sides of ① by X^θ, we get

$$X^\theta(AX) = X^\theta(\lambda X)$$

$$X^\theta A X = X^\theta \lambda X \text{ —— } ②$$

Taking transposed conjugate both sides, we get.

$$(X^\theta A X)^\theta = (X^\theta \lambda X)^\theta$$

$$X^\theta A^\theta (X^\theta)^\theta = X^\theta \bar{\lambda} (X^\theta)^\theta$$

$$X^\theta A^\theta X = \bar{\lambda} X^\theta X$$

$$X^\theta A X = \bar{\lambda} X^\theta X \text{ —— } ③ \quad [\because A^\theta = A]$$

From ② and ③, we get

$$\lambda X^\theta X = \bar{\lambda} X^\theta X$$

$$(\lambda - \bar{\lambda}) X^\theta X = 0$$

$$\lambda - \bar{\lambda} = 0$$

$$\lambda = \bar{\lambda}$$

$\therefore \lambda$ is real

$\left[\begin{array}{l} \because X \text{ is non zero vector} \\ X^\theta \text{ " " "} \\ X X^\theta \text{ " " "} \\ \text{i.e } X X^\theta \neq 0 \end{array}\right.$

$\therefore$ Hence the eigen values of a hermitian matrix are real.

Eg verify that the eigen values of hermitian matrix $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ are real.

sol: Given that $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$ i.e $\begin{vmatrix} 4-\lambda & 1-3i \\ 1+3i & 7-\lambda \end{vmatrix} = 0$.

$$(4-\lambda)(7-\lambda) - 10 = 0$$

$$\lambda^2 - 11\lambda + 18 = 0$$

$$\lambda = 2, 9$$

The Eigen values of A are $\lambda = 2, 9$

Which are real

$\therefore$ The Eigen values of hermitian matrix A are real.

<u>Theorem</u>:- The Eigen values of a skew hermitian matrix are either purely imaginary or zero.

<u>Proof</u>:- Let A be a skew hermitian matrix i.e $A^\theta = -A$. and λ be the eigen value of A.

We prove that $\lambda = 0$ or λ is an imaginary.

If λ is an eigen value of A and x be the corrosponding eigen vector then $AX = \lambda X$ —— (1)

Pre multiply bothsides of (1) by 'i', we get

$$i(AX) = i(\lambda X)$$

$$(iA)X = (i\lambda)X.$$

By definition, $i\lambda$ is an eigen value of iA

Since A is skew hermitian, we have $A^\theta = -A$

$\Rightarrow$ iA is hermitian.

Since $(iA)^\theta = -iA^\theta$

$$= (-i)(-A)$$

$$(iA)^\theta = iA$$

A is skew hermitian then iA is hermitian matrix —— (2)

From (1) and (2), We have $i\lambda$ is the eigen value of a hermitian matrix iA.

Since we know that Eigen values of a hermitian matrix are real. 16

$\therefore$ $i\lambda$ is real number

i.e λ is zero or purely imaginary.

Hence the Eigen values of a skew hermitian matrix are either purely imaginary or zero.

Eg: Verify that an eigen values of skew hermitian matrix $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ are either purely imaginary or zero.

Sol: Given that $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$

$$\text{i.e} \quad \begin{vmatrix} 3i-\lambda & 2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$$

$$(3i-\lambda)(-i-\lambda) - (2+i)(-2+i) = 0$$

$$3 - 3i\lambda + i\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 - 2i\lambda + 8 = 0$$

$$\lambda = \frac{2i \pm \sqrt{-4-32}}{2} = \frac{2i \pm 6i}{2} = i \pm 3i$$

$$\lambda = 4i, -2i$$

The Eigen values of A are $\lambda = 4i, -2i$

Which are purely imaginary.

$\therefore$ The Eigen values of given skew hermitian matrix are purely imaginary.

<u>Theorem</u>:- The Eigen values of unitary matrix is of unit modulus.

<u>Proof</u>:- Let A be a unitary matrix i.e $AA^\theta = I = A^\theta A$ and λ be the Eigen value of A

We prove that $|\lambda| = 1$.

If λ is an eigen value of A and x be the corrospording eigen vector then $AX = \lambda X$ —— ①.

Taking transposed conjugate on bothsides of ①, we get

$$(AX)^\theta = (\lambda X)^\theta$$

$$X^\theta A^\theta = \bar{\lambda} X^\theta \quad \text{——} ②$$

Multiplying ① and ②, we get

$$(X^\theta A^\theta)(AX) = (\bar{\lambda} X^\theta)(\lambda X)$$

$$X^\theta (A^\theta A) X = \lambda\bar{\lambda}(X^\theta X)$$

$$X^\theta I X = \lambda\bar{\lambda}(X^\theta X) \qquad \left[\begin{array}{l} \because AA^\theta = I = A^\theta A \\ A \text{ is unitary} \end{array}\right.$$

$$X^\theta X = \lambda\bar{\lambda}(X^\theta X)$$

$$(1-\lambda\bar{\lambda}) X^\theta X = 0$$

$$1-\lambda\bar{\lambda} = 0 \qquad \left[\begin{array}{l} \because X \text{ is non zero vector} \\ X^\theta \text{ is non zero vector} \\ XX^\theta \\ \text{i.e } XX^\theta \neq 0 \end{array}\right.$$

$$\lambda\bar{\lambda} = 1$$

$$|\lambda|^2 = 1$$

$$|\lambda| = 1$$

Hence the Eigen values of a unitary matrix are of unit modulus.

Eg Verify that the eigen values of a unitary matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ are of unit modulus

Sol: Given that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$

$$\text{i.e} \begin{vmatrix} \frac{1}{\sqrt{2}}-\lambda & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}}-\lambda \end{vmatrix} = 0$$

$$-\left(\frac{1}{\sqrt{2}}+\lambda\right)\left(\frac{1}{\sqrt{2}}-\lambda\right) - \frac{1}{2} = 0.$$

$$-\left(\frac{1}{2}-\lambda^2\right) - \frac{1}{2} = 0.$$

$$\lambda^2 - 1 = 0$$

$\lambda = \pm 1$, Eigen values of A are $\lambda = 1, -1$

$$|\lambda| = 1$$

$\therefore$ Eigen values of unitary matrix A are of unit modulus.

Theorem :- An Eigen values of Idempotent matrix are 0 and 1

Proof :- Let A be an Idemponent matrix i.e $A^2 = A$. —— (1)

Let λ be an eigen value of A and X is corresponding eigen vector. Then

$$AX = \lambda X \text{ —— (2)}.$$

We prove that an eigen values of A are 0 and 1 i.e $\lambda = 0$ and 1.

We know that If λ is an eigen value of A corresponding to the eigen vector X then λ^n is an eigen value of A^n corresponding to the eigen vector X.

We have $A^n X = \lambda^n X$

$$\Rightarrow A^2 X = \lambda^2 X \text{ —— (3)}$$

From (2) and (3), we get

$$AX = \lambda^2 X \text{ —— (4)}.$$

From (2) and (4), we get

$$\lambda^2 X = \lambda X$$

$$(\lambda^2 - \lambda) X = 0.$$

$$\lambda^2 - \lambda = 0 \quad [\because X \neq 0]$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = 0, \lambda = 1$$

$\therefore$ An Eigen values of Idempotent matrix A are 0 and 1.

QUADRATIC FORMS

Quadratic form :—

A homogeneous polynomial of second degree in n variables $x_1, x_2, x_3 \dots x_n$ is called a quadratic form in the n variables.

It is denoted by Q.

Thus $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ is a quadratic form in n variables $x_1, x_2, \dots x_n$

[OR]

An expression of the form $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ where a_{ij}'s are elements of a field F is called a quadratic form in n variables. $x_1, x_2, x_3 \dots x_n$ over a field F.

If a_{ij}'s belongs to a real number field R then the above quadratic form is said to be a "real quadratic form" in n variables $x_1, x_2, \dots x_n$

It is denoted by Q i.e $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$

Eg:- (i) $Q = x^2$ is a quadratic form in a single variable x.

(ii) $Q = 3x^2 + 4xy + 7y^2$ is a quadratic form in two variables x, y.

(iii) $Q = x^2 + y^2 + 3z^2 + 4xy - 7xz + 8yz$ is a quadratic form in 3 variables.

Quadratic form corrosponding to a Real Symmetric Matrix :—

Let $A = [a_{ij}]_{n\times n}$ be a real symmetric matrix and let $X = [x_1\ x_2\ x_3 \dots x_n]^T$ be a column matrix Then $X^T A X$ will determine a quadratic form

$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ ..

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j = a_{11}x_1^2 + a_{12}x_1x_2 + \dots + a_{1n}x_1x_n + a_{21}x_2x_1 + a_{22}x_2^2 + \dots + a_{2n}x_2x_n + \dots + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \dots + a_{nn}x_n^2.$$

$$= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + \dots + (a_{1n} + a_{n1})x_1x_n + a_{22}x_2^2 + (a_{23} + a_{32})x_2x_3 + \dots + (a_{2n} + a_{n2})x_2x_n + \dots + a_{nn}x_n^2$$

Matrix of a Quadratic form :-

If $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ is a quadratic form in n variables $x_1, x_2, \dots x_n$ over a field F then there exists a unique symmetric matrix A of order n such that $Q = X^T A X$

Where $X = [x_1 \; x_2 \; x_3 \; \dots \; x_n]^T$

Here the symmetric matrix A is called the matrix of the quadratic form Q.

Find the quadratic form relating to the symmetric matrix $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 4 \end{bmatrix}$

sol: Let $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 4 \end{bmatrix}$

The Quadratic form related to the given matrix is $X^T A X$.

Where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $X^T = [x_1 \; x_2 \; x_3]$.

$\therefore$ Required quadratic form $= X^T A X = [x_1 \; x_2 \; x_3] \begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= [x_1 \; x_2 \; x_3] \begin{bmatrix} x_1 + 2x_2 + 6x_3 \\ 2x_1 + x_2 + 3x_3 \\ 6x_1 + 3x_2 + 4x_3 \end{bmatrix}$

$= x_1(x_1 + 2x_2 + 6x_3) + x_2(2x_1 + x_2 + 3x_3) + x_3(6x_1 + 3x_2 + 4x_3)$

$= x_1^2 + x_2^2 + 4x_3^2 + 4x_1x_2 + 6x_2x_3 + 12x_1x_3$.

Write down the ~~qua~~ symmetric matrix. of the quadratic form.

$2x_1^2 + 3x_2^2 + 44x_3^2 - 3x_1x_2 + 4x_2x_3 - 5x_1x_3$.

sol:- Given that $2x_1^2 + 3x_2^2 + 44x_3^2 - 3x_1x_2 + 4x_2x_3 - 5x_1x_3$.

It can be written as $2x_1^2 + 3x_2^2 + 44x_3^2 - \frac{3}{2}x_1x_2 - \frac{3}{2}x_2x_1 + 2x_2x_3 + 2x_3x_2 - \frac{5}{2}x_1x_3 - \frac{5}{2}x_3x_1$

$\therefore$ The Matrix of quadratic form $A = \begin{bmatrix} 2 & -3/2 & -5/2 \\ -3/2 & 3 & 2 \\ -5/2 & 2 & 44 \end{bmatrix}$

Linear Transformation of a Quadratic form :-

Let $Q = X^T A X$ be a quadratic form in n variables $x_1, x_2, x_3 \dots x_n$ and the symmetric matrix $A = [a_{ij}]_{n \times n}$ be the matrix of Q.

Let $X = PY$ be a non singular transformation when P is a non singular matrix of order n and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$

$$X = PY$$

$$X^T = (PY)^T = Y^T P^T$$

$$\begin{aligned} Q &= X^T A X \\ &= (PY)^T A (PY) \\ &= Y^T (P^T A P) Y \end{aligned}$$

$$Q = Y^T B Y \qquad \text{where } B = P^T A P$$

$$\begin{aligned} B^T &= (P^T A P)^T \\ &= P^T A^T (P^T)^T \\ &= P^T A P \end{aligned}$$

$$B^T = B$$

$\therefore$ B is symmetric.

Hence $Y^T B Y$ is another quadratic form in n variables $y_1, y_2, y_3 \dots y_n$

Thus the linear transformation $X = PY$ transforms the given quadratic form Q to another quadratic form $Q' = Y^T B Y$.

i.e $Y^T B Y$ is the linear transform of $X^T A X$ under the linear transform $X = PY$.

If P is a non singular matrix of order n, then the linear transformation $X = PY$ is said to be a non singular linear transformation

A non singular transformation is also called regular transformation

If P is an orthogonal matrix of order n then the linear transformation $X = PY$ is called an orthogonal transformation. (2)

Canonical form or Normal form of a quadratic form :-

A real quadratic form in which the product terms are missing and which contains only terms of squares of variables is called a canonical form.

Eg:- $Q = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + \cdots + a_n x_n^2$ is a canonical form.

[OR]

If $X^T A X$ is a real quadratic form in n variables, then there exists a real non singular linear transformation $X = PY$ which transforms $X^T A X$ to the form $y_1^2 + y_2^2 + y_3^2 + \cdots + y_r^2$.

This expression is called the canonical form or normal form of the given quadratic form $X^T A X$.

Rank of a Quadratic form :-

Let $X^T A X$ be a quadratic form over a field F. The rank of the matrix A is called the rank of the quadratic form $X^T A X$.

Working procedure for the reduction of Quadratic form to the Normal form or Canonical form :-

Let $Q = X^T A X$ be a quadratic form of n-variables.

Let A be the matrix of the ~~matrix~~ quadratic form.

Here A is the symmetric matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Step(i) :- We can write $A_{3\times3} = I_3 A I_3$ —— (1)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now reduce the matrix A on the L.H.S to the diagonal form by applying a finite no. of elementary transformations. Each row transformation will be applied to the pre factor I_3 and each column transformation applied to the post factor I_3 on the R.H.S of eqn(1)

Step(ii):- If $a_{11} \neq 0$ then by using a_{11} position make a_{21}, a_{31} positions as zero. The same row operations will be applied pre factor of A on R.H.S.

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} & \times & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step(iii):- By using a_{11} position make a_{12}, a_{13} positions as zero. The same column operations will be applied post factor of A on R.H.S.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & a''_{23} \\ 0 & a''_{32} & a''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & \times & \\ & & \\ & & \end{bmatrix}$$

Step(iv):- If $a''_{22} \neq 0$ then by using a''_{22} position make a''_{32} position as zero. The same row operation will be applied prefactor of A on R.H.S

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & a''_{23} \\ 0 & 0 & a'''_{33} \end{bmatrix} = \begin{bmatrix} & \times & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Step (v):- By using a''_{22} position make a''_{23} position as zero. The same column operations will be applied post factor of A on R.H.S

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & 0 \\ 0 & 0 & a^{iv}_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

The Resulting equation is $P^T A P$ = Diagonal matrix.

Where P is a non singular matrix of order n.

Step (vi) :- Finally we can interpret the above result interms of quadratic forms. (4)

If $x^T A x$ be a real quadratic form in n variables then there exists a linear transformation $x = PY$ where P is a non singular matrix of order n, transforms the quadratic form $x^T A x$ to a diagonal form.

i.e $Y^T P^T A P Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \cdots + \lambda_r y_r^2$

i.e a sum of r - square terms. Here r gives the rank of the quadratic form $x^T A x$.

Note :- In the above procedure of diagonal form if we make the diagonal elements as 1 or -1 or 0 then we obtain the required canonical form or normal form of the given quadratic form.

Index of the quadratic form :-

Let $y_1^2 + y_2^2 + y_3^2 + \cdots + y_p^2 - y_{p+1}^2 + \cdots + y_r^2$ be a canonical form or normal form of real quadratic form $x^T A x$. The number of positive terms in the normal form of $x^T A x$ is called the index of the quadratic form.

It is denoted by s.

The number of non positive terms is equal to $r - s$.

Signature of the quadratic form :-

The difference of the number of positive terms and the non-positive terms is called the signature of the quadratic form.

$\therefore$ Signature $= s - (r - s) = 2s - r$.

Nature of a Quadratic form :-

(5)

The quadratic form X^TAX in n variables is said to be

(i) Positive Definite :- If $r=n$ and $s=n$ (OR) If all the eigen values of A are positive.

(ii) Nagative Definite :- If $r=n$ and $s=0$ (OR) If all the eigen values of A are -ve.

(iii) Positive semi definite :- If $r<n$ and $s=r$ [OR] If all the eigen values of A are ≥ 0 and atleast one eigen value is zero.

(iv) Nagative semi definite :- If $r<n$ and $s=0$ [OR] If all the eigen values of $A \leq 0$ and atleast one eigen value is zero.

(v) Indefinite :- In all other cases [OR] If A has positive as well as negative eigen values.

1) Identify Nature, Index, Rank and signature of the quadratic form $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$.

Sol:- The given quadratic form can be written as $x_1^2+4x_2^2+x_3^2-2x_1x_2-2x_2x_1$
$+x_1x_3+x_3x_1-2x_2x_3-2x_3x_2$

The matrix of the quadratic form is $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 1\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

$$\text{i.e } \begin{vmatrix}1-\lambda & -2 & 1\\ -2 & 4-\lambda & -2\\ 1 & -2 & 1-\lambda\end{vmatrix}=0$$

$R_1 \to R_1+R_2+R_3$

$$\begin{vmatrix}-\lambda & -\lambda & -\lambda\\ -2 & 4-\lambda & -2\\ 1 & -2 & 1-\lambda\end{vmatrix}=0$$

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$$-\lambda\begin{vmatrix}1 & 1 & 1\\ -2 & 4-\lambda & -2\\ 1 & -2 & 1-\lambda\end{vmatrix}=0$$

$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$-\lambda\begin{vmatrix}1 & 0 & 0\\ -2 & 6-\lambda & 0\\ 1 & -3 & -\lambda\end{vmatrix}=0$$

$$-\lambda\left[-\lambda(6-\lambda)-0\right]=0$$

$$\lambda = 0, 0, 6.$$

$\therefore$ The Eigen values of A are $\lambda = 0, 0, 6$.

(i) The Nature of the quadratic form is positive semi definite.

(ii) The Index of the quadratic form is 1.

(iii) Rank of the quadratic form is 1.

(iv) Signature of the ~~to~~ quadratic form is $2s - r = 1$.

Find the transformation which will transform $4x^2+3y^2+z^2-8xy-6yz+4zx$ into a sum of squares and find the reduced form.

sol:- Given that the quadratic form

$$4x^2+3y^2+z^2-8xy-6yz+4zx.$$

It can be written as

$$4x^2+3y^2+z^2-4xy-4yx-3yz-3zy+2zx+2xz.$$

The matrix of the quadratic form is

$$A = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix}$$

We write $A = I_3 A I_3$.

We apply elementary operations on A of L.H.S and we apply the same row operations on the pre factor and column operations on the post factor

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2+R_1 \quad R_3 \to 2R_3+R_1$

$$\begin{bmatrix} 4 & -4 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2+C_1 \quad C_3 \to 2C_3-C_1$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$C_2 \longrightarrow 3C_2 + C_1 \qquad C_3 \longrightarrow 3C_3 - C_1$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & -3 \\ 0 & -3 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$R_3 \longrightarrow 7R_3 + R_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & -3 \\ 0 & 0 & 144 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$C_3 \longrightarrow 7C_3 + C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 1008 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix}$$

This is of the form $B = P^T A P$.

$$B = \text{Diag}\{6, 21, 1008\} = P^T A P$$

Thus the non singular linear transformation $X = PY$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & -21 \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ transforms the given quadratic form to

to the diagonal form which is given by $6y_1^2 + 21y_2^2 + 1008y_3^2$.

Rank of the quadratic form $r = 3$.

Index of the quadratic form $s = 3$.

Signature of the quadratic form $2s - r = 6 - 3 = 3$.

$\therefore$ The given quadratic form is positive definite.

$\therefore$ The required nonsingular linear transformation which brings about an diagonal form is $X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & -21 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$x = y_1 + y_2 - 6y_3 \qquad y = 3y_2 + 3y_3, \quad z = -21y_3$

QUADRATIC FORMS

I: Reduce the following quadratic forms into a sum of squares. Indicate the nature, rank, index and signature of the quadratic form. Also write the corresponding linear transformation which brings about the normal reduction.

R.NO	Q.NO
1-20	1-3
21-40	4-6
41-60	7-10

(i) $3x^2 + 3y^2 + 3z^2 + 4xy + 8xz + 8yz$.

Ans:- Rank = 3, Index = 2, Nature : Indefinite.

(ii) $4x^2 + 3y^2 + z^2 - 8xy - 6yz + 4zx$.

Ans:- Rank = 3, Index = 2, Nature : Indefinite.

(iii) $5x^2 + 26y^2 + 10z^2 + 4yz + 14zx + 6xy$.

Ans:- Rank = 2, Index = 2 Nature : Positive semi definite.

(iv) $7x^2 + 6y^2 + 5z^2 - 4xy - 4yz$

Ans:- Rank = 3, Index = 3. Nature : Positive definite.

(v) $3x^2 + 2y^2 + z^2 + 4xy - 2xz + 6yz$

Ans:- Rank = 3, Index = 2 (Nature : Indefinite.

(vi) $-3x^2 - 3y^2 - 3z^2 - 2xy - 2yz + 2xz$

Ans:- Rank = Index = Nature : Nagative definite.

(vii) $6x^2 + 17y^2 + 3z^2 - 20xy - 14yz + 8zx$

Ans:- Rank = 2 Index = 2 Nature : Positive semi definite.

(viii) $6x^2 + 3y^2 + 14z^2 + 4yz + 18xz + 4xy$.

Ans:- Rank = 3 Index = 3 Nature : Positive definite.

(ix) $4x_1^2 + 9x_2^2 + 2x_3^2 + 8x_2x_3 + 6x_3x_1 + 6x_1x_2$

Ans:- Rank = 3 Index = 2 Nature Indefinite

(x) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3$.

Ans:- Rank = 3 Index = 3 Nature : Positive definite.

II. Reduce the following quadratic forms to canonical form. In each case find the matrix of the transform. Also find rank, index, nature and signature of the quadratic form.

R.NO	Q.NO
1–20	1–3
21–40	4–6
41–60	7–10

(i) $2xy + 2yz + 2zx$

Ans:- Rank = 3, Index = 1 Nature: Indefinite.

(ii) $2x^2 + 2y^2 + 2z^2 - 2xy + 2xz - 2yz$

Ans:- Rank = 3 Index = 3 Nature Positive definite.

(iii) $3x^2 + 3z^2 + 4xy + 8xz + 8yz$

Ans:- Rank = 3, Index = 1 Nature: Indefinite.

(iv) $x^2 + 4y^2 + z^2 + 4xy + 6yz + 2zx$

Ans:- Rank = Index = Nature:

(v) $x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx - 4xy - 2xt - 6zt$.

Ans:- Rank = 3 Index = 2 Nature: Indefinite.

(vi) $2x^2 + y^2 - 3z^2 + 12xy - 4zx - 8yz$.

Ans:- Rank = 3 Index = 1. Nature: Indefinite.

(vii) $x_1^2 + 3x_2^2 + 5x_3^2 - 4x_1x_2 + 2x_3x_1 + 4x_2x_3$

Ans:- Rank = 3 Index = 2 Nature: Indefinite.

(viii) $2xy - 4yz - 6zx$

Ans:- Rank = Index = Nature:

(ix) $9x^2 + 2y^2 + 2z^2 + 6xy + 2yz - 2zx$

Ans:- Rank = 3 Index = 3 Nature: Positive definite.

(x) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$.

Ans:- Rank = 3 Index = 3 Nature: Positive definite.

Reduction of the Quadratic form to canonical form by Orthogonal Transformation :-

(1)

If in the transformation $X = PY$, P is an orthogonal matrix and if $X = PY$ transforms the Quadratic form Q to the canonical form the Q is said to be reduced to the canonical form by an orthogonal transformation.

Working procedure :-

Let $Q = X^T A X$ be a given quadratic form.

Step1 :- Let A be the matrix of the quadratic form.

Step2 :- The characteristic equation of A is $|A - \lambda I| = 0$

Solve the characteristic equation and find the eigen values $\lambda_1, \lambda_2, \lambda_3$ of the matrix A.

Step3 :-

case(i) :- If the eigen values $\lambda_1, \lambda_2, \lambda_3$ of the matrix A are distinct.

step(i) :- Find the eigen vectors X_1, X_2, X_3 corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$ and these eigen vectors are linearly independent. observe that these eigen vectors are pairwise orthogonal.

$\therefore$ The matrix A is diagonalizable.

step(ii) :-

$$\text{Modal Matrix} = [X_1 \;\; X_2 \;\; X_3] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

step(iii) :- Construct the normalized eigen vectors e_1, e_2, e_3 corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$.

Where $e_1 = \dfrac{X_1}{\|X_1\|}$ $\quad e_2 = \dfrac{X_2}{\|X_2\|}$ $\quad e_3 = \dfrac{X_3}{\|X_3\|}$

$\|X_1\| = \sqrt{a_1^2 + b_1^2 + c_1^2}$ $\quad \|X_2\| = \sqrt{a_2^2 + b_2^2 + c_2^2}$ $\quad \|X_3\| = \sqrt{a_3^2 + b_3^2 + c_3^2}$

Step (iv):- Define the normalized modal matrix.

$$P = [e_1 \; e_2 \; e_3] = \left[\frac{X_1}{\|X_1\|} \; \frac{X_2}{\|X_2\|} \; \frac{X_3}{\|X_3\|}\right]$$

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This P will be an orthogonal matrix.

By definition of an orthogonal matrix.

$$PP^T = P^TP = I$$

$$\Longrightarrow P^{-1} = P^T$$

Step (v):- Find $P^{-1}AP$ (OR) P^TAP

Which is the diagonal matrix of A.

$$P^{-1}AP = P^TAP = D = \mathrm{Diag}[\lambda_1 \; \lambda_2 \; \lambda_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Step (vi):- Now Define the orthogonal transformation $X = PY$

Which transforms the given Quadratic form $Q = X^TAX$ to the normal form is given by $Q = X^TAX$ Where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$Q = (PY)^T A (PY)$$

$$= (Y^TP^T) A (PY)$$

$$= Y^T (P^TAP) Y$$

$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$Q = Y^TDY$$

$$Q = [y_1 \; y_2 \; y_3]^T \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

Which is the required Normal form (OR) Canonical form.

Case (ii):- If the eigen values $\lambda_1 \; \lambda_2 \; \lambda_3$ of the matrix A are not distinct. If suppose λ_1 is repeated two times.

Step (i):- Find the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$ and these eigen vectors are linearly independent.

If Algebraic multiplicity of an eigen value $\lambda \neq$ Geometric multipli-city of an eigen value λ then Diagonalization for the matrix A is not possible.

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So we stop the procedure.

else (Algebraic multiplicity of an eigen value λ = Geometric multiplicity of an eigen value λ)

$\therefore$ The matrix A is diagonalizable.

goto step(ii).

step(ii):- Here we observe that the eigen vectors X_1, X_2 are not pairwise orthogonal corresponding to the eigen value λ_1.

Now we find the eigen vector X_1 is pairwise orthogonal to X_2 and X_3

Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is pairwise orthogonal to X_2 and X_3.

Where $X_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ $X_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$

X_1, X_2 are pairwise orthogonal if $x_1 a_2 + y_1 b_2 + z_1 c_2 = 0$

X_1, X_3 are pairwise orthogonal if $x_1 a_3 + y_1 b_3 + z_1 c_3 = 0$.

Solve the above equations, we get the values of x_1, y_1 and z_1

$\therefore$ The Eigen vectors X_1, X_2 and X_3 are pairwise orthogonal.

step(iii):- Modal Matrix $= [X_1 \; X_2 \; X_3] = \begin{bmatrix} x_1 & a_2 & a_3 \\ y_1 & b_2 & b_3 \\ z_1 & c_2 & c_3 \end{bmatrix}$

step(iv):- Construct the normalized eigen vectors e_1, e_2, e_3 corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$.

$\|X_1\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$ $\quad \|X_2\| = \sqrt{a_2^2 + b_2^2 + c_2^2}$ $\quad \|X_3\| = \sqrt{a_3^2 + b_3^2 + c_3^2}$

Where $e_1 = \dfrac{X_1}{\|X_1\|}$ $\quad e_2 = \dfrac{X_2}{\|X_2\|}$ $\quad e_3 = \dfrac{X_3}{\|X_3\|}$

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step (v) :- Normalized Modal Matrix

$$P = [e_1 \ e_2 \ e_3] = \left[\frac{X_1}{\|X_1\|} \ \frac{X_2}{\|X_2\|} \ \frac{X_3}{\|X_3\|} \right]$$

This P will be an orthogonal matrix.

By definition of an orthogonal matrix $PP^T = P^TP = I \Rightarrow P^{-1} = P^T$

step (vi) :- Find $P^{-1}AP$ (or) P^TAP

Which is the diagonal matrix of A.

$$P^{-1}AP = P^TAP = D = \text{Diag}[\lambda_1 \ \lambda_2 \ \lambda_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Now define an orthogonal transformation $X = PY$ which transforms the given Quadratic form $Q = X^TAX$ to the normal form is given by.

$$Q = X^TAX \qquad \text{where } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

$$= (PY)^TA(PY)$$

$$= (Y^TP^T)A(PY)$$

$$= Y^T(P^TAP)Y$$

$$Q = Y^TDY$$

$$= [y_1 \ y_2 \ y_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2.$$

Which is the required normal form or canonical form.

Pairwise Orthogonal :—

Let A be a square matrix (symmetric) of order 3.

If $\lambda_1 \ \lambda_2 \ \lambda_3$ are three distinct eigen values of A then the corresponding eigen vectors X_1, X_2 and X_3 are pairwise orthogonal.

$$\text{Let } X_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \quad X_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \quad X_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$$

X_1, X_2 are pairwise orthogonal if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

X_2, X_3 are pairwise orthogonal if $a_2a_3 + b_2b_3 + c_2c_3 = 0$

X_1, X_3 are pairwise orthogonal if $a_1a_3 + b_1b_3 + c_1c_3 = 0$

(1) Reduce the quadratic form $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$ to the normal form by orthogonal transformation.

Sol:- Given that $Q = 3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$.

The above quadratic form can be written as

$$Q = 3x^2 + 2y^2 + 3z^2 - xy - yx - yz - zy$$

The matrix of the quadratic form is

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$$(3-\lambda)\left[(2-\lambda)(3-\lambda) - 1\right] + 1\left[(-1)(3-\lambda)\right] = 0$$

$$(3-\lambda)\left[6 - 3\lambda - 2\lambda + \lambda^2 - 1 - 1\right] = 0$$

$$(3-\lambda)(\lambda^2 - 5\lambda + 4) = 0$$

$$(3-\lambda)(\lambda-4)(\lambda-1) = 0$$

$$\lambda = 1, 3, 4.$$

The eigen values of A are $\lambda = 1, 3, 4$.

These eigen values are distinct.

$\therefore$ The matrix A is diagonalizable.

Now the Eigen vector corresponding to the Eigen values λ are obtained by solving the system of equations $(A - \lambda I)X = 0$.

i.e $\begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case(i): - Eigen vector corresponding to the Eigen value $\lambda = 3$: —

For $\lambda = 3$, The system (1) can be written as

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the coeff. matrix into echelon form by applying E-row operations only

$R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Rank of the coefficient matrix is 2 i.e $\rho(A) = 2$ = The No. of non zero rows equivalent to A.

So that the homogeneous system has $n - r = 3 - 2 = 1$ L.I solution.

There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = 3$.

To determine this, we have to assign an arbitrary value for one variable.

From the above system the linear equations are

$$x + y + z = 0$$

$$y = 0.$$

$$x + z = 0$$

choose $x = k_1$

$z = -x = -k_1$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ -k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is the eigen vector corrosponding to the eigen value $\lambda = 3$.

Case (ii) :- Eigen vector corrosponding to the eigen value $\lambda = 1$:-

For $\lambda = 1$, The System (1) can be written as

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the coeff matrix into echelon form by applying E-row operations only

$R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + 2R_1$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 =$ The No. of Non zero rows equivalent to A

So that the homogeneous system has $n - r = 3 - 2 = 1$ L.I solution.

There is only one linearly independent eigen vector corrosponding to the eigen value $\lambda = 1$.

To determine this, we have to assign an arbitrary value for one variable.

From the above System the linear equations are

$$-x + y - z = 0$$

$$y - 2z = 0.$$

$\therefore$ This matrix P will reduce the matrix A to be diagonal form which is given by $P^{-1}AP = D$

i.e $P^{T}AP = D$

$$D = P^{T}AP = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus the orthogonal transformation $X = PY$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$ transforms the given quadratic

form to the normal form is given by.

$$Q = X^{T}AX$$

$$Q = (PY)^{T}A(PY)$$

$$Q = Y^{T}(P^{T}AP)Y$$

$$Q = Y^{T}DY$$

$$Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = 3y_1^2 + y_2^2 + 4y_3^2$$

$\therefore$ The required orthogonal transformation which brings about the normal form is given by $X = PY$ i.e $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$x = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3$, $y = \frac{2}{\sqrt{6}}y_2 - \frac{1}{\sqrt{3}}y_3$ $z = \frac{-1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3$

The Rank of the Q.F $r = 3$, Index of the Q.F $= s = 3$

signature of the Q.F $= 2s - r = 3$ Nature of Q.F is +ve definite.

Reduce the quadratic form $3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$ into sum of squares form by an orthogonal transformation and give the matrix of transformation.

Sol:- Given that $3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$

The given Q.F can be written as

$$3x_1^2+3x_2^2+3x_3^2+x_1x_2+x_2x_1+x_1x_3+x_3x_1-x_2x_3-x_3x_2$$

The matrix of the quadratic form is

$$A=\begin{bmatrix}3&1&1\\1&3&-1\\1&-1&3\end{bmatrix}$$

The characteristic equation of A is $|A-\lambda I|=0$

$$\text{i.e}\begin{vmatrix}3-\lambda&1&1\\1&3-\lambda&-1\\1&-1&3-\lambda\end{vmatrix}=0$$

$R_1\rightarrow R_1+R_3$

$$\begin{vmatrix}4-\lambda&0&4-\lambda\\1&3-\lambda&-1\\1&-1&3-\lambda\end{vmatrix}=0$$

$$(4-\lambda)\begin{vmatrix}1&0&1\\1&3-\lambda&-1\\1&-1&3-\lambda\end{vmatrix}=0.$$

$R_3\rightarrow R_3-R_2$

$$(4-\lambda)\begin{vmatrix}1&0&1\\1&3-\lambda&-1\\0&\lambda-4&4-\lambda\end{vmatrix}=0.$$

$$(4-\lambda)^2\begin{vmatrix}1&0&1\\1&3-\lambda&-1\\0&-1&1\end{vmatrix}=0.$$

$C_3 \longrightarrow C_3 - C_1$

$$(4-\lambda)^2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 3-\lambda & -2 \\ 0 & -1 & 1 \end{vmatrix} = 0$$

$$(4-\lambda)^2 \left[(3-\lambda) - 2\right] = 0 .$$

$$(4-\lambda)^2 \cdot (\lambda+1) = 0$$

$$\lambda = 1, 4, 4 .$$

The eigen values of A are $\lambda = 4, 4, 1$.

The algebraic multiplicities of an eigen values 4 and 1 are 2 and 1.

Now we have to find the eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corrosponding to

the eigen values λ are obtained by solving the system of equations

$(A - \lambda I)X = 0$ i.e. $\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ —— (1)

Case(i):- Eigen vector corrousponding to the eigen value $\lambda = 1$:—

For $\lambda = 1$, The system (1) can be written as

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and determine the rank of the matrix.

$R_2 \longrightarrow 2R_2 - R_1 \quad R_3 \longrightarrow 2R_3 - R_1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

$$R_3 \longrightarrow R_3 + R_2$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Rank of the coefficient matrix $r = 2 =$ The No. of non zero rows.

So that the system have $n - r = 3 - 2 = 1$ L.I solution.

There is only one L.I eigen vector corresponding to the eigen value. $\lambda = 1$.

To determine this, we have to assign an arbitrary value for $n - r = 3 - 2 = 1$ variables.

The linear equations are

$$2x_1 + x_2 + x_3 = 0$$

$$3x_2 - 3x_3 = 0 \Longrightarrow x_2 - x_3 = 0.$$

Choose $x_3 = k_1$

$$x_2 = x_3 = k_1$$

$$2x_1 = -x_2 - x_3 = -2k_1$$

$$x_1 = -k_1$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{where } k_1 \neq 0$$

$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$. So that the geometric multiplicity of $\lambda = 1$ is 1.

Case (ii):- Eigen vector corresponding to the eigen value $\lambda = 4$:-

For $\lambda = 4$, The system (1) can be written as

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and determine the rank of the coefficient matrix.

$$R_2 \longrightarrow R_2 + R_1, \quad R_3 \longrightarrow R_3 + R_1$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Rank of the coefficient matrix $r = 1$ = No. of non zero rows.

So that the system have $n - r = 3 - 1 = 2$ L.I solutions.

There are two linearly independent eigen vectors corresponding to the eigen value $\lambda = 4$.

To determine this, we have to assign an arbitrary value for $n - r = 3 - 1 = 2$ variables

The linear equation is

$$-x_1 + x_2 + x_3 = 0$$

Choose $x_2 = k_2$

$x_3 = k_3$

$$x_1 = x_2 + x_3 = k_2 + k_3$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 + k_3 \\ k_2 \\ k_3 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent eigen vectors corresponding to the eigen value $\lambda = 4$.

So that the ~~algebraic~~ geometric multiplicity of an eigen value $\lambda = 4$ is 2.

Since the geometric multiplicity of each eigen value of A coincides with the algebraic multiplicity

$\therefore$ A is a diagonalizable matrix.

Now we observe that the eigen vectors x_2 and x_3 are not pair-wise orthogonal.

Now we have to find the another linearly independent eigen vector x_2 of A corresponding to the eigen value $\lambda = 4$ such that x_1, x_2 and x_2, x_3 are pairwise orthogonal.

Let $x_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the another L.I eigen vector corresponding to the eigen value $\lambda = 4$.

x_1, x_2 are pairwise orthogonal if $-a + b + c = 0$ —— (2)

x_2, x_3 are pairwise orthogonal if $a + 0 \cdot b + c = 0$ —— (3)

Solving (2) and (3), we get

1	1	-1	1
0	1	1	0

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{-1}$$

$x_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 4$ and is orthogonal to x_1 and x_3.

Now the eigen vectors $x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are pair-wise orthogonal.

$$\text{Modal matrix} = [x_1 \ x_2 \ x_3] = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$\|x_1\| = \sqrt{1+1+1} = \sqrt{3}$ $\quad \|x_2\| = \sqrt{1+4+1} = \sqrt{6}$

$\|x_3\| = \sqrt{1+0+1} = \sqrt{2}$

Normalized modal matrix $P = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \end{bmatrix}$

$$P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Here P is an orthogonal matrix.

By def. $PP^T = P^TP = I$

$$\Longrightarrow \quad P^{-1} = P^T$$

This matrix P will reduce the matrix A to the diagonal form which is given by $P^{-1}AP = D$ i.e $P^TAP = D$.

$$P^TAP = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is the spectral matrix.

Thus the orthogonal transformation $X = PY$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -1/\sqrt{6} & \frac{1}{\sqrt{2}} \end{bmatrix}$ transforms the given quadratic form to

canonical form is given by

$$Q = X^TAX$$

$$Q = (PY)^T A (PY)$$

$$= Y^T (P^T A P) Y$$

$$Q = Y^T D Y$$

$$Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = y_1^2 + 4y_2^2 + 4y_3^2$$

Rank of the Quadratic form $r = 3$

Index of the Quadratic form $s = 3$.

Signature of the Quadratic form $2s - r = 6 - 3 = 3$.

Nature of the Quadratic form is positive definite.

$\therefore$ The required orthogonal transformation which brings about the normal form is given by $X = PY$.

i.e
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \frac{-1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{6}} y_2 + \frac{1}{\sqrt{2}} y_3$$

$$x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_2$$

$$x_3 = \frac{1}{\sqrt{3}} y_1 - \frac{1}{\sqrt{6}} y_2 + \frac{1}{\sqrt{2}} y_3 .$$

Maximize and Minimize the quadratic form $Q = X^T A X$ subject to $x^2+y^2+z^2 = 1$:-

Let $Q = X^T A X$ be the quadratic form.

step (i) :- Write the matrix of the given quadratic form.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ which is the symmetric matrix.

step (ii) :- The characteristic equation of A is $|A - \lambda I| = 0$

i.e $$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0.$$

Solve the characteristic equation, we get the eigen values of A.

step (iii) :- The eigen values of the matrix A are $\lambda_1, \lambda_2, \lambda_3$.

case (i) :- Let $\lambda = \max\{\lambda_1, \lambda_2, \lambda_3\}$

Suppose $\lambda = \lambda_1$

Find the eigen vector corresponding to the eigen value $\lambda = \lambda_1$

Let $X_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$

Find the normalized eigen vector $e_1 = \frac{X_1}{\|X_1\|}$

$$\|X_1\| = \sqrt{a_1^2 + b_1^2 + c_1^2}$$

$$e_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}} \\ \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}} \\ \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}} \end{bmatrix}$$

Substitute the normalized eigen vector in given quadratic form,

we get maximum value of Q.

$\therefore$ Maximum value of Q = Maximum eigen value $= \lambda_1$.

<u>Case(ii)</u>:- For minimize the quadratic form. Q.

Let $\lambda = \min\{\lambda_1, \lambda_2, \lambda_3\}$

Suppose $\lambda = \lambda_2$

Find the eigen vector x_2 corresponding to the eigen value. $\lambda = \lambda_2$

Let $x_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$

Find the normalized eigen vector $e_2 = \dfrac{x_2}{\|x_2\|}$

$$\|x_2\| = \sqrt{a_2^2 + b_2^2 + c_2^2}$$

$$e_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} \dfrac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \\ \dfrac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \\ \dfrac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \end{bmatrix}$$

Substitute the normalized eigen vector in given quadratic form.

we get manimum value of Q.

$\therefore$ Minimum value of Q = Minimum eigen value $= \lambda_2$.

Nature of a Quadratic form $Q = X^TAX$ with the help of principal minors of the matrix A :—

The nature of a quadratic form can be determined from a study of the principal minors of the matrix of the quadratic form. In this method, the quadratic form need not be put in the canonical form.

Principal minors :—

Let $A = [a_{ij}]$ be a square matrix of order n. Then

$$M_1 = |a_{11}| \quad M_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad M_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdots \quad M_n = |A|.$$

Working Rules :—

Case (i) :- A real quadratic form Q is positive definite if and only if all the principal minors of A are positive i.e $M_i > 0$ for all $i \le n$.

Eg:- $Q = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1$

The matrix A of the given quadratic form is given by

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$M_1 = |1| = 1 > 0 \qquad M_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = 1 - \frac{1}{4} = \frac{3}{4} > 0.$$

$$M_3 = |A| = 1\left(1-\frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}-\frac{1}{2}\right) = \frac{3}{4} - \frac{1}{8} - \frac{1}{8} = \frac{1}{2} > 0.$$

$$M_3 = > 0.$$

$$\therefore M_i > 0 \quad \forall i \le 3.$$

$\therefore$ The nature of the given quadratic form is positive definite.

Case (ii):- A real quadratic form Q is negative definite if and only if $M_1, M_3, M_5 \ldots$ are all negative and $M_2, M_4, M_6 \ldots$ are all positive.

i.e $(-1)^i M_i > 0$ for all i.

Eg:- $Q = -4x^2 - 2y^2 - 13z^2 - 4xy - 8yz - 4xz$.

The matrix A of the given quadratic form is given by

$$A = \begin{bmatrix} -4 & -2 & -2 \\ -2 & -2 & -4 \\ -2 & -4 & -13 \end{bmatrix}$$

$M_1 = |-4| = -4 < 0$ $\quad$ $M_2 = \begin{vmatrix} -4 & -2 \\ -2 & -2 \end{vmatrix} = 8 - 4 = 4 > 0.$

$M_3 = |A| = \left[-4(26-16) + 2(26-8) - 2(8-4)\right] = -40 + 36 - 8 = -12 < 0.$

Here $M_1 < 0$, $M_2 < 0$ $\quad M_2 > 0$.

$\therefore$ The nature of given quadratic form is negative definite.

Case (iii):- If some of the principal minors in case (i) are zero while the others are positive then the quadratic form Q is positive semi definite i.e $M_i \geq 0 \ \forall i \leq n$ and at least one $M_i = 0$.

Eg.- $Q = 10x^2 + 2y^2 + 5z^2 + 6yz - 10zx - 4xy$.

The matrix A of the given quadratic form is given by.

$$A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

$M_1 = |10| = 10 > 0.$ $\quad$ $M_2 = \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} = 20 - 4 = 16 > 0.$

$M_3 = |A| = 10[10-9] + 2[-10+15] - 5[-6+10] = 10 + 10 - 20 = 0$

Here $M_1 > 0$, $M_2 > 0$ and $M_3 = 0$

$\therefore$ The given quadratic form is positive semi definite.

nature of

Case (iv) :- If some of the principal minors in case (ii) are zero then Q is negative semi definite.

i.e. $(-1)^i M_i \geq 0 \ \forall i \leq n$ and at least one $M_i = 0$.

Eg :- $Q = -3x_1^2 - 3x_2^2 - 7x_3^2 - 6x_1x_2 - 6x_2x_3 - 6x_3x_1$.

The matrix of the quadratic form is

$$A = \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -7 \end{bmatrix}$$

$M_1 = |-3| = -3 < 0 \qquad M_2 = \begin{vmatrix} -3 & -3 \\ -3 & -3 \end{vmatrix} = 0$

$M_3 = |A| = -3[21-9] + 3[21-9] - 3[9-9] = 0$.

Here $M_1 < 0 \quad M_2 = 0 \quad M_3 = 0$.

$\therefore$ The nature of given quadratic form Q is negative semi definite.

Case (v) :- In all other cases, Q is indefinite.

Eg:- $Q = x^2 + 4y^2 + 4z^2 + 4xy + 6xz + 16yz$

The matrix of the quadratic form is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 4 \end{bmatrix}$$

$M_1 = |1| > 0 \quad M_2 = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$.

$M_3 = |A| = 1(16 - \cancel{64}) - 2(8-24) + 3(16-12) = -48 + 32 + 12 = -8 < 0$

Here $M_1 > 0 \quad M_2 = 0 \quad M_3 < 0$.

$\therefore$ The nature of the given quadratic form is indefinite.

Reduce the following quadratic forms to canonical form by an orthogonal transformation. Indicate its nature, rank, index and signature of the quadratic form. Also write the corresponding linear transformation which brings about the normal form.

(i) $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$

Ans:- Rank $=3$, Index $=3$. Nature : Positive definite.

Eigen values : $1, 2, 4$.

(ii) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$.

Ans:- Rank $=3$, Index $=3$. Nature Positive definite.

Eigen values : $2, 3, 6$

(iii) $3x^2 - 2y^2 - z^2 + 12yz + 8zx - 4xy$.

Ans:- Rank $=3$ Index $=2$ Nature: Indefinite.

Eigen values : $3, 6, -9$

(iv) $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4zx$.

Ans:- Rank $=2$ Index $=2$ Nature : Positive semi definite.

Eigen values : $0, 3, 15$

(v) $3x^2 + 2y^2 + 3z^2 - 2xy + 2yz$.

Ans:- Rank $=3$. Index $=3$ Nature : Positive definite

Eigen values. $3, 1, 4$.

(vi) $7x^2 + 5y^2 + 6z^2 - 4xz - 4yz$

Ans. Rank $=3$, Index $=3$ Nature : Positive definite.

Eigen values : $3, 6, 9$.

(vii) $3x^2 + 2y^2 - 4xz$. Eigen values : $-1, 2, 4$

Ans:- Rank $=3$, Index $=2$ Nature Indefinite.

(viii) $6x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ Eigen values : $6, 2, 4$

Ans:- Rank $=3$, Index $=3$ Nature Positive definite.

Reduce the following quadratic forms to canonical form by an ortho-gonal transformation. Indicate Rank, index, nature and signature of the quadratic form. Also indicate the matrix of the transformation

(i) $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz + 2zx$.

Ans:- Rank $= 3$ Index $= 3$ Nature: Positive definite.

Eigen values: 1, 1, 4

(ii) $2xy + 2yz + 2zx$

Ans: Rank $= 3$ Index $= 1$ Nature: Indefinite.

Eigen values: $-1, -1, 4$.

(iii) $3x^2 + 3y^2 + 3z^2 + 2xy + 2xz - 2yz$

Ans:- Rank $= 3$, Index $= 3$ Nature: Positive definite.

Eigen values: 1, 4, 4

(iv) $2x_1x_2 + 2x_1x_3 - 2x_2x_3$.

Ans:- Rank $= 3$, Index $= 2$ Nature: Indefinite.

Eigen values 1, 1, -2

(v) $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$, Eigen values: 2, 2, 8.

Ans:- Rank $= 3$ Index $= 3$ Nature Positive definite.

(vi) $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

Ans: Rank $= 2$ Index $= 2$ Nature: Positive semi definite.

Eigen values: 0, 3, 3

(vii) $-3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

Ans:- Rank $= 3$, Index $= 0$ Nature: Nagative definite.

Eigen values: $\cancel{-4}, -4, -1$.

(viii) $x^2 + y^2 + z^2 + 4yz + 4xy + 4zx$.

Ans.- Rank $= 3$, Index $= 2$ Nature: Indefinite.

(1) Find the maximum and minimum values of $f(x,y) = 3x^2 - 3y^2 + 8xy$ subject to $x^2 + y^2 = 1$.

2

Ans:- Max. of $f = 5$, Min. of $f = -5$.

(2) Find the maximum and minimum values of $f(x,y,z) = 3x^2 + 3z^2 + 2y^2 + 2xz$ subject to $x^2 + y^2 + z^2 = 1$.

Ans:- Max. of $f = 4$. Min. of $f = 2$

(3) Find the maximum and minimum values of $f(x,y,z) = 10x^2 + 2y^2 + 5z^2 - 4xy - 10xz + 6yz$ subject to $x^2 + y^2 + z^2 = 1$.

Max. of $f = 14$. Min. of $f = 0$.

(4) Find the maximum and minimum values of $2x^2 + 5y^2 + 3z^2 + 4xy$. subject to $x^2 + y^2 + z^2 = 1$.

Max. of $f = 6$ Min. of $f = 1$.

(1) Identify the nature of the following quadratic forms. Also write Rank, Index and signature of the quadratic form.

2

(a) $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$.

Ans:- Nature: +ve semi definite, Index = 1, Rank = 1.

(b) $x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$.

Ans:- Nature: Indefinite Index = 1 Rank = 2.

(c) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$.

Ans:- Nature: Positi definite Index = 3 Rank = 3.

(d) $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz$.

Ans:- Nature: +ve semi definite. Index = 2 Rank = 2.

Reduce the following quadratic forms to canonical form by Lagrange's method. Also write the corresponding linear transformation. Find its rank, index nature and signature of the quadratic form.

1(a) $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

Ans: Rank = 3, Nature - Indefinite Index = 2.

1(b) $2x_1^2 + 7x_2^2 + 5x_3^2 - 8x_1x_2 - 10x_2x_3 + 4x_1x_3$.

Ans: Rank = 3 Nature : Indefinite Index = 2.

1(c) $x_1^2 + 3x_2^2 + x_3^2 + 2x_1x_2 + 4x_2x_3 + 6x_1x_3$.

Ans:- Rank = 3 Nature : Indefinite Index = 2.

1(d) $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$.

Ans:- Rank = 1. Nature : Positive semi definite Index = 1.

(e) $x_1^2 + 6x_2^2 + 18x_3^2 + 4x_1x_2 + 8x_1x_3 - 4x_2x_3$.

Ans: Rank = 3 Nature : Indefinite Index = 2.

(f) $x^2 + y^2 + z^2 - 2xy + 4xz + 4yz$

Ans:- Rank = 3 Nature : Indefinite Index = 2.

1(g) $x_1^2 - 4x_2^2 + 5x_3^2 + 2x_1x_2 - 4x_1x_3 + 2x_4^2 - 6x_3x_4$

Ans: Rank = 4 Nature : Indefinite Index = 2.

1(h) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

Ans:- Rank = 3 Nature : Positive definite Index = 3.

R.NO	Q.NO
1 - 20	1 - 3
21 - 40	4 - 6
41 - 60	6 - 8

MODULE -III

DIFFERENTIAL CALCULUS

Neighbourhood of a point :-

If $a \in R$ and $\varepsilon > 0$ then the set $\{x \in R / |x-a| < \varepsilon\}$ is called ε-neighbourhood of 'a' in R.

$$\varepsilon\text{-nbd of } a = \{x \in R / |x-a| < \varepsilon\}$$
$$= \{x \in R / -\varepsilon < x-a < \varepsilon\}$$
$$= \{x \in R / a-\varepsilon < x < a+\varepsilon\}$$
$$= (a-\varepsilon, a+\varepsilon), \text{ an open interval.}$$

ε-nbd of 'a' is denoted as $N_\varepsilon(a)$ or $N(\varepsilon, a)$.

Note :- ε-nbd of a point p is the set of all points which are within ε-distance of p on either side.

Eg :- $(2-\frac{1}{2}, 2+\frac{1}{2}) = (\frac{3}{2}, \frac{5}{2})$ is $\frac{1}{2}$-nbd of 2.

Deleted ε-nbd of $a = \{x \in R / |x-a| < \varepsilon, x \neq a\}$
$$= \{x \in R / 0 < |x-a| < \varepsilon\}$$
$$= (a-\varepsilon, a) \cup (a, a+\varepsilon)$$

Deleted ε-nbd of 'a' is denoted as $N_\varepsilon(a) - \{a\}$

Limit of a function :-

Let $f : S \rightarrow R$ be a function 'a' be a limit point of an aggregate S and $l \in R$. The function f tends to limit l as x tends to a if for each $\varepsilon > 0$ then there exists $\delta > 0$ such that $x \in S$ and $0 < |x-a| < \delta$

$\Rightarrow |f(x) - l| < \varepsilon$

We write $f(x) \rightarrow l$ as $x \rightarrow a$ or $\underset{x \to a}{Lt} f(x) = l$.

$\underset{x \to a^-}{Lt} f(x) = l$ is called limit from below or left hand limit of the function.

$\underset{x \to a^+}{Lt} f(x) = l$ is called limit from above or right hand limit of the function.

$\underset{x \to a}{Lt} f(x) = l$ is called limit of the function.

Continuity of a function at a point :-

Let s be an aggregate $f : s \to R$ be a function and $a \in s$. f is said to be continuous at 'a' if given $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in s$, $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Definition (Limit Notation of Continuity at a point) :-

Let $f : s \to R$ be a function and $a \in s$ be a limit point of s.

f is said to be continuous at 'a' from left if $\underset{x \to a^-}{Lt} f(x) = f(a)$.

f is said to be continuous at 'a' from right if $\underset{x \to a^+}{Lt} f(x) = f(a)$

f is said to be continuous at 'a' if $\underset{x \to a}{Lt} f(x) = f(a)$.

Note :- (i) $f(x)$ is continuous at $x = a \Rightarrow$ There is no break in the graph of $y = f(x)$ in a nbd of the point $(a, f(a))$.

(ii) $f(x)$ is continuous on $[a, b] \Rightarrow$ The graph of $y = f(x)$ is unbroken from the point $(a, f(a))$ to the point $(b, f(b))$.

(iii) If f, g are continuous at 'a', $a \in R$ then $f + g$ is continuous at 'a' and $f - g$ is continuous at a.

(iv) If f, g are continuous at a then fg is continuous at a.

(v) If g is continuous at 'a' and $g(a) \neq 0$ then $\frac{1}{g}$ is continuous at a.

(vi) If f, g are continuous at 'a' and $g(a) \neq 0$ then $\frac{f}{g}$ is continuous at 'a'.

(vii) The constant function $f(x) = k$, $k \in R$ is continuous on R.

(viii) The trigonometric function $\sin x$ and $\cos x$ are continuous on R.

(ix) The function $f(x) = e^x$, $x \in R$ is continuous on R.

(x) The function $f(x) = \log x$, $x \in R^+$ is continuous on R^+

(xi) The function $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_0, a_1, a_2 \cdots a_n \in R$ and $n \in N$ is continuous on R (OR) Every polynomial function is continuous on R.

→ Check whether the function $f(x) = \begin{cases} x^2+1 & , 0 \le x < 1 \\ 3-x & , 1 \le x \le 2 \end{cases}$ is continuous.

Sol:- Given that $f(x) = \begin{cases} x^2+1 & 0 \le x < 1 \\ 3-x & 1 \le x \le 2 \end{cases}$

→ $f(x) = x^2+1$ is continuous in $0 \le x < 1$ since $f(x)$ is polynomial.

→ $f(x) = 3-x$ is continuous in $1 \le x \le 2$ since $f(x)$ is polynomial.

At $x = 1$, $f(1) = 3-1 = 2$

$$\underset{x \to 1^-}{Lt} f(x) = \underset{x \to 1^-}{Lt} x^2+1 = 2$$

$$\underset{x \to 1^+}{Lt} f(x) = \underset{x \to 1^+}{Lt} 3-x = 2$$

$$\therefore \underset{x \to 1}{Lt} f(x) = f(1)$$

$\therefore$ $f(x)$ is continuous at $x = 1$.

$\therefore$ $f(x)$ is continuous in $[0, 2]$.

Derivability of a function at a point :-

Let S be an aggregate and $f : S \rightarrow R$ be a function and let $c \in S$ be a limit point of S.

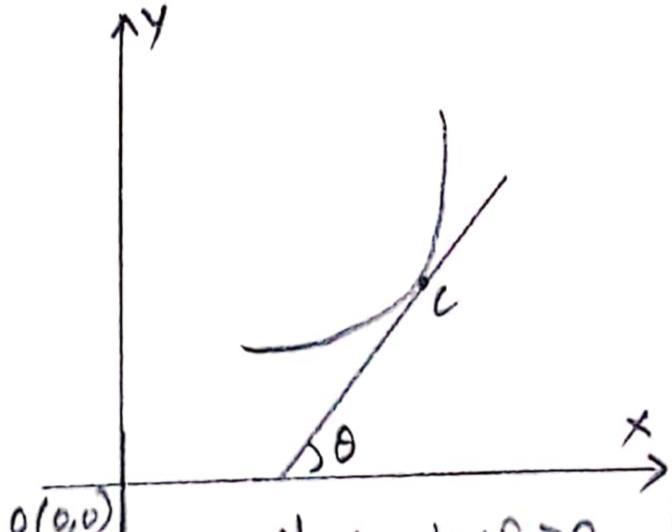
If $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ $(x \neq c)$ exists then we say that f is derivable at 'c'. The limit is called the derivative of f at c and is denoted by $f'(c)$.

→ Geometrically the derivative $f'(c)$ represents the slope of the tangent line at $(c, f(c))$ to the graph $y = f(x)$

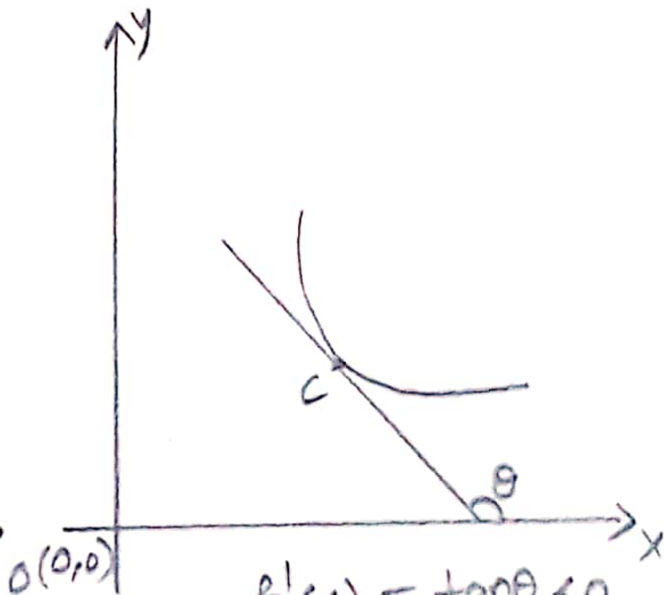
If $f(x)$ is derivable in $[a,b]$ there exists a unique tangent to the curve at every point in the interval $[a,b]$.

If $f'(c)$ is positive it means that $f(x)$ is an increasing function as x increases via c.

If $f'(c)$ is negative it means that $f(x)$ is decreasing function as x increases via c.



$f'(c) = \tan\theta > 0$



$f'(c) = \tan\theta < 0$

$f'(c)$ is the slope of the tangent to the curve $y = f(x)$ at $x = c$.

→ Check whether the function $f(x) = \begin{cases} x^2 - 2 & -1 \le x < 0 \\ x - 2 & 0 \le x \le 1 \end{cases}$ is derivable at $x = 0$.

Sol: Given that $f(x) = \begin{cases} x^2 - 2 & -1 \le x < 0 \\ x - 2 & 0 \le x \le 1 \end{cases}$

→ Left hand derivative at $x = 0$

$$f'(0^-) = \underset{x \to 0^-}{Lt} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0^-}{Lt} \frac{(x^2 - 2) - (-2)}{x}$$

$$= \underset{x \to 0^-}{Lt} \frac{x^2 - 2 + 2}{x} = \underset{x \to 0^-}{Lt} \frac{x^2}{x} = 0.$$

→ Right hand derivative at $x = 0$

$$f'(0^+) = \underset{x \to 0^+}{Lt} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0^+}{Lt} \frac{(x - 2) - (-2)}{x}$$

$$= \underset{x \to 0^+}{Lt} \frac{x}{x} = 1$$

$$f'(0^-) \neq f'(0^+).$$

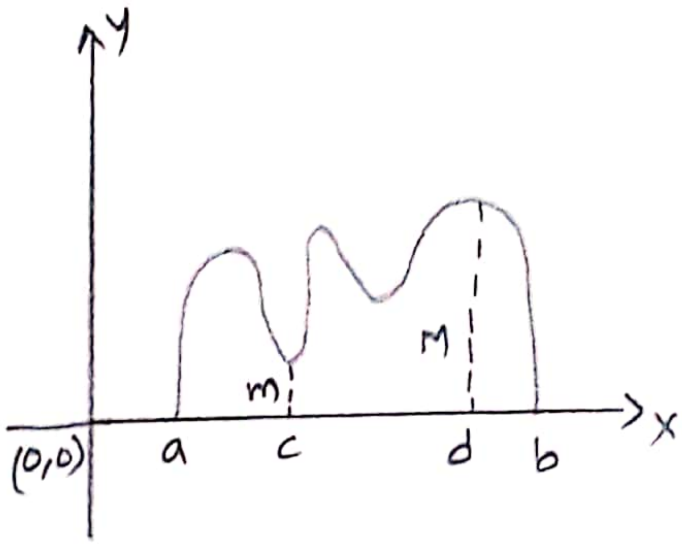
∴ The function is not derivable at $x = 0$.

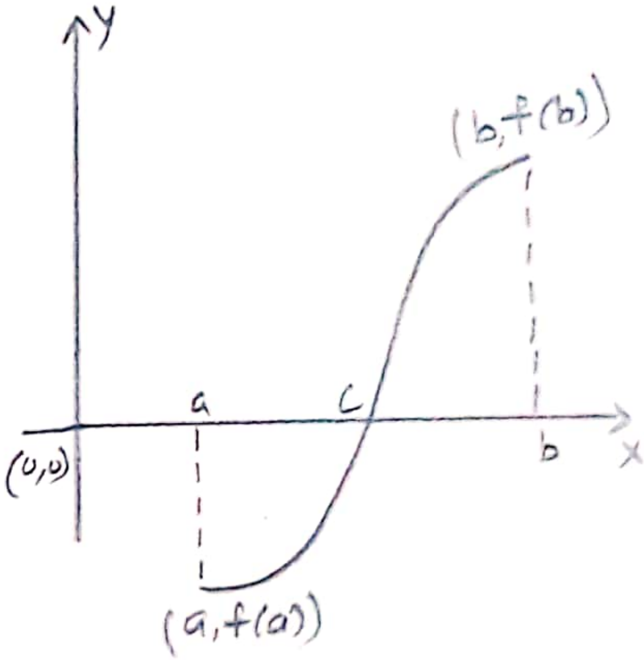
Properties of continuous function :-

1) If $f(x)$ is continuous in $[a,b]$ $f(x)$ is bounded in $[a,b]$. Also it attains greatest lower bound and least upper bound. If m is the greatest lower bound and M is the least upper bound of $f(x)$ in $[a,b]$ there exists points c and d in $[a,b]$ such that $f(c)=m$ and $f(d)=M$.

2) If $f(x)$ is continuous in $[a,b]$ it attains all values between $f(a)$ and $f(b)$

3) If $f(x)$ is continuous in $[a,b]$ and $f(a)$, $f(b)$ are of opposite signs then there exists atleast one point $c\in(a,b)$ such that $f(c)=0$





Rolle's Theorem :—

If a function $f: [a,b] \to R$ is such that

(i) f is continuous on $[a,b]$

(ii) f is derivable on (a,b).

(iii) $f(a) = f(b)$ then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Geometrical Interpretation of Rolle's Theorem :—

Let $f: [a,b] \to R$ be a function satisfying the three conditions of Rolle's theorem, then

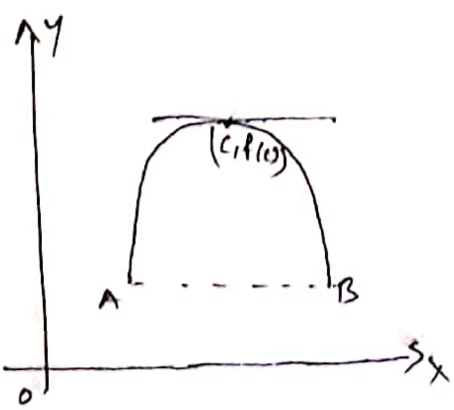
(i) The curve $y = f(x)$ is continuous in $[a,b]$ (That is from the point $A(a, f(a))$ to the point $B(b, f(b))$.

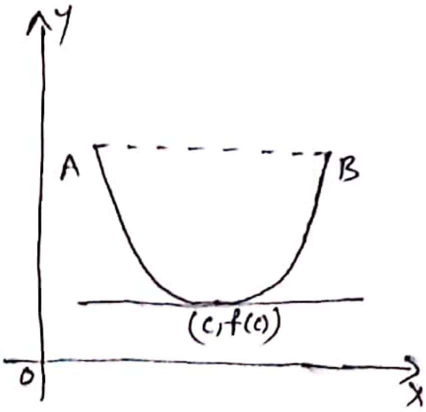
(ii) At every point $x = c$ where $a < c < b$ at the point $(c, f(c))$ on the curve $y = f(x)$ there is a unique tangent to the curve.

(iii) $f(a) = f(b)$ i.e the two endpoints of the curve $y = f(x)$ corresponding to $x = a$, $x = b$ have the same ordinate.

By Rolle's theorem there is atleast one $c \in (a,b)$ such that $f'(c) = 0$

Therefore there is atleast one point $c(c, f(c))$ between A and B on the curve at which the tangent line is parallel to the x-axis.





(1) verify Rolles theorem for $f(x) = \log\left[\frac{x^2+ab}{(a+b)x}\right]$ in $[a,b]$

sol: Given that $f(x) = \log\left[\frac{x^2+ab}{(a+b)x}\right]$ in $[a,b]$.

We know that logarithm function is continuous on R^+

(i) $f(x) = \log \frac{x^2+ab}{(a+b)x}$ is continuous on $[a,b]$ $[\because [a,b] \subseteq R^+]$

(ii) $f(x) = \log\left[\frac{x^2+ab}{(a+b)x}\right]$

$$= \log(x^2+ab) - \log(a+b)x.$$

$$f(x) = \log(x^2+ab) - \log(a+b) - \log x.$$

Diff w.r.t 'x', we get

$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$, $f'(x)$ exists on (a,b)

(ii) $\therefore f$ is derivable on (a,b).

(iii) $f(x) = \log\left[\frac{x^2+ab}{(a+b)x}\right]$

$$f(a) = \log \frac{a^2+ab}{(a+b)a} = 0$$

$$f(b) = \log \frac{b^2+ab}{(a+b)b} = 0$$

$$f(a) = f(b).$$

The function f satisfies all the conditions of Rolles theorem.

$\therefore$ By Rolles theorem

There exists atleast one point $c \in (a,b)$ such that $f'(c) = 0$.

i.e. $\frac{2c}{c^2+ab} - \frac{1}{c} = 0.$

$$\frac{2c^2 - c^2 - ab}{c(c^2+ab)} = 0.$$

$$c^2 - ab = 0$$

$$c^2 = ab$$

$$c = \pm\sqrt{ab}$$

$$c = \sqrt{ab} \in (a, b).$$

$\therefore$ $c = \sqrt{ab}$ such that $f'(c) = 0$ and $a < c < b$.

Hence Rolles Theorem verified

(2) verify Rolles theorem for the function $f(x) = \frac{\sin x}{e^x}$ (or) $e^{-x}\sin x$ in $[0, \pi]$.

sol:- Given that $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$.

We know that the function ~~f(x)~~ $\sin x$ is continuous on R.

Then the function $\sin x$ is continuous on $[0, \pi]$ $(\because [0,\pi] \subseteq R)$

We know that the function e^x is continuous on R.

Then the function e^x is continuous on $[0,\pi]$ $(\because [0,\pi] \subseteq R)$

and $e^x \neq 0$ $\forall x \in [0,\pi]$

(i) $\therefore$ $f(x) = \frac{\sin x}{e^x}$ is continuous on $[0,\pi]$.

(ii) $f(x) = \frac{\sin x}{e^x}$

Diff. w.r.t 'x', we get

$$f'(x) = \frac{e^x \cos x - \sin x\, e^x}{(e^x)^2} = \frac{\cos x - \sin x}{e^x}$$

$\longrightarrow$ $f'(x)$ exists $\forall x \in (0,\pi)$.

f is derivable on $(0,\pi)$.

(iii) $f(x) = \frac{\sin x}{e^x}$

$f(0) = \frac{\sin 0}{e^0} = 0$

$f(\pi) = \frac{\sin \pi}{e^\pi} = 0$

$\therefore f(0) = f(\pi)$

The function $f(x) = \frac{\sin x}{e^x}$ satisfies all three conditions of Rolles theorem.

By Rolles Theorem

There exists at least one point $c \in (0, \pi)$ such that $f'(c) = 0$.

i.e $\frac{\cos c - \sin c}{e^c} = 0$

$\cos c - \sin c = 0$

$\cos c = \sin c$

$\frac{\sin c}{\cos c} = 1$

$\tan c = 1$

$c = \frac{\pi}{4}$.

$\therefore c = \frac{\pi}{4} \in (0, \pi)$

$\therefore c = \frac{\pi}{4} \in (0, \pi)$ such that $f'(c) = 0$ and $0 < c < \pi$

Hence Rolles theorem verified.

(3) Verify Rolles theorem for the function $f(x) = (x-a)^m (x-b)^n$ where m, n are positive integers, in $[a, b]$.

Sol: Given that $f(x) = (x-a)^m (x-b)^n$ in $[a, b]$

(i) We know that every polynomial fun is continuous on R

$\cdot f(x) = (x-a)^m (x-b)^n$ is continuous on $[a, b]$ $(\because [a, b] \subseteq R)$

(ii) $f(x) = (x-a)^m (x-b)^n$

Diff w.r.t 'x', we get.

$$f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1}$$

$$= (x-a)^{m-1}(x-b)^{n-1}\left[m(x-b)+n(x-a)\right]$$

$$f'(x) = (x-a)^{m-1}(x-b)^{n-1}\left[(m+n)x-(mb+na)\right].$$

$f'(x)$ exists $\forall x \in (a,b)$

$\therefore$ f is derivable on (a,b)

(iii) $f(x) = (x-a)^m (x-b)^n$

$f(a) = (a-a)^m (a-b)^n = 0$

$f(b) = (b-a)^m (b-b)^n = 0.$

$\therefore$ $f(a) = f(b)$.

The function $f(x) = (x-a)^m (x-b)^n$ satisfies all the conditions Rolle's theorem.

$\therefore$ ~~There exist~~ By Rolles Theorem.

There exists at least one point $c \in (a,b)$ such that $f'(c) = 0$

i.e $(c-a)^{m-1}(c-b)^{n-1}\left[(m+n)c-(mb+na)\right] = 0$

$$(m+n)c - (mb+na) = 0$$

$$c = \frac{mb+na}{m+n}$$

$$c = \frac{mb+na}{m+n} \in (a,b)$$

$\therefore$ $c = \frac{mb+na}{m+n} \in (a,b)$ such that $f'(c) = 0$ and $a < c < b$.

$\therefore$ Hence Rolles Theorem verified

It is given that the Rolle's theorem holds for the function $f(x) = x^3 + bx^2 + cx$
$1 \le x \le 2$ at the point $x = \frac{4}{3}$. Find the values of b and c.

Sol:- Given that $f(x) = x^3 + bx^2 + cx$, in $1 \le x \le 2$.

(i) We know that Every polynomial function is continuous in R.
The given function $f(x) = x^3 + bx^2 + cx$ is continuous in $[1,2]$ $(\because [1,2] \subseteq R)$

(ii) $f'(x) = 3x^2 + 2bx + c$.

f is derivable in $(1,2)$

(iii) $f(1) = 1 + b + c$, $f(2) = 8 + 4b + 2$.

We have $f(1) = f(2)$

$1 + b + c = 8 + 4b + 2$

$3b + c + 7 = 0$ —— ①

By Rolles theorem, There exists a point $x \in (1,2)$ such that $f'(x) = 0$

i.e $3x^2 + 2bx + c = 0$

We have $x = \frac{4}{3}$, $3 \cdot \frac{16}{9} + 2b \cdot \frac{4}{3} + c = 0$

$8b + 3c + 16 = 0$ —— ②

Solving equations ① and ②, we get

$b = -5$, $c = 8$.

(4) Verify whether can we apply Rolle's theorem for the function $f(x) = |x|$ in $-1 \leq x \leq 1$

Sol:- Given that $f(x) = |x|$ in $[-1, 1]$.

We know that $f(x) = |x|$

i.e $f(x) = x$ for $x \geq 0$

$= -x$ for $x < 0$

(i) We know that $f(x) = |x|$ is continuous on R.

$\therefore$ $f(x) = |x|$ is continuous on $[-1, 1]$ $(\because [-1, 1] \subseteq R)$

(ii) $f(x) = |x|$ is not derivable at $x = 0$.

We have $f(0) = |0| = 0$.

$$L.H.D = Lf'(0) = \underset{x \to 0^-}{Lt} \frac{f(x) - f(0)}{x - 0}$$

$$= \underset{x \to 0^-}{Lt} \frac{|x| - 0}{x} = \underset{x \to 0^-}{Lt} \frac{-x}{x} =.$$

$$= \underset{x \to 0^-}{Lt} (-1) = -1.$$

$$Lf'(0) = -1$$

$$R.H.D = Rf'(0) = \underset{x \to 0^+}{Lt} \frac{f(x) - f(0)}{x - 0}$$

$$= \underset{x \to 0^+}{Lt} \frac{|x| - 0}{x} = \underset{x \to 0^+}{Lt} \frac{x}{x}$$

$$= \underset{x \to 0^+}{Lt} 1 = 1.$$

since $Lf'(0) \neq Rf'(0)$

$\therefore$ f is not derivable at $x = 0$.

$\therefore$ $f(x)$ is not derivable in $(-1, 1)$ at $x = 0$.

Hence Rolle's theorem is not applicable at $f(x) = |x|$ in $[-1, 1]$.

Algebraic Interpretation of Rolle's Theorem :-

Let $f(x)$ be a polynomial in x. If $f(x)=0$ satisfies all the conditions of Rolle's Theorem and $x=a$, $x=b$ be the roots of the equation $f(x)=0$ then atleast one root of the equation $f'(x)=0$ lies between a and b.

Prove that the equation $2x^3-3x^2-x+1=0$ has atleast one root between 1 and 2.

Sol:- Let $f'(x) = 2x^3-3x^2-x+1$

Let $f(x) = \int f'(x)\,dx = \frac{x^4}{2} - x^3 - \frac{x^2}{2} + x$.

$f(x) = \frac{x^4}{2} - x^3 - \frac{x^2}{2} + x$ in $[1,2]$.

(i) We know that every polynomial function is continuous on R.

Since $[1,2] \subseteq R$.

$\therefore$ The function $f(x) = \frac{x^4}{2} - x^3 - \frac{x^2}{2} + x$ is continuous on $[1,2]$.

(ii) $f(x) = \frac{x^4}{2} - x^3 - \frac{x^2}{2} + x$.

$f'(x) = 2x^3 - 3x^2 - x + 1$.

$f'(x)$ exists $\forall x \in (1,2)$

f is derivable on $(1,2)$.

(iii) At $x=1$, $f(1) = 0$

At $x=2$ $f(2) = 0$.

$\therefore$ $f(1) = f(2)$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolles theorem.

$\therefore$ By Rolle's theorem, There exists at least one point $c \in (1,2)$ Such that $f'(c)=0$ i.e $2c^3-3c^2-c+1=0$.

$c = -0.618,\ 1.618,\ 0.5$

$\therefore$ $c = 1.618 \in (1,2)$

This shows that c is the root of the equation $2x^3-3x^2-x+1=0$ which lies between 1 and 2.

LAGRANGE'S MEAN VALUE THEOREM :—

Theorem :- Let $f: [a,b] \longrightarrow R$ be a function such that

(i) it is continuous in $[a,b]$

(ii) it is differentiable in (a,b).

Then there exists at least one point c in (a,b) such that

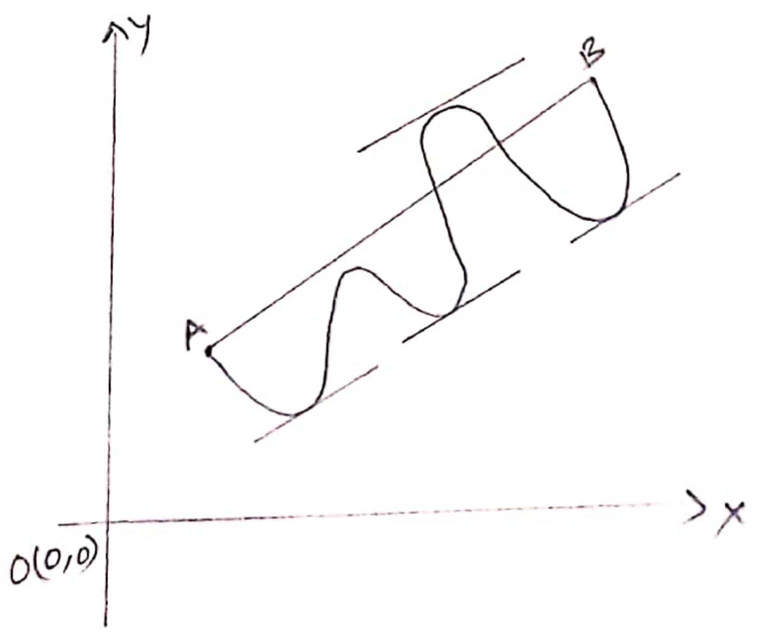
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

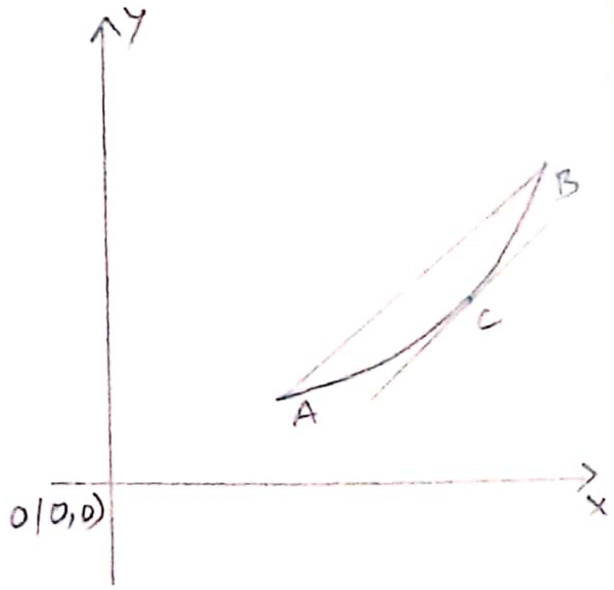
Geometric interpretation of Lagrange's mean value Theorem :—

(i) The curve $y=f(x)$ is continuous in $[a,b]$.

(ii) At every point $x=c$, where $a<c<b$, at the point $(c,f(c))$ on the curve $y=f(x)$ there is a unique tangent to the curve.

Then Lagrange's mean value theorem says that there is atleast one point on the curve where the tangent to the curve is parallel to the chord joining the end points $A(a,f(a))$ and $B(b,f(b))$ on the curve since the slope at c, $f'(c)$ is equal to the slope of the chord $AB = \dfrac{f(b)-f(a)}{b-a}$.





Alternate form of the Lagrange's Mean value Theorem :—

In Lagrange's mean value theorem put $b = a+h$ so that $h = b-a$

Any point $x = c$ in (a, b) i.e in $(a, a+h)$ will be of the form $c = a + \theta h$

for some θ lying between 0 and 1.

Further $\dfrac{f(b)-f(a)}{b-a} = \dfrac{f(a+h)-f(a)}{h}$

$$\text{Now } f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow f'(a+\theta h) = \frac{f(a+h)-f(a)}{h}$$

This can be rewritten as $f(a+h) = f(a) + h f'(a+\theta h)$

Lagrange's mean value theorem can be stated alternately as below.

Let $f(x)$ be (i) Continuous in $[a, a+h]$ (ii) differentiable in $(a, a+h)$

Then there exists a positive real number θ, $0<\theta<1$. Such that

$f(a+h) = f(a) + h f'(a+\theta h)$.

<u>Note</u>:- (i) In some problems, we may have to find $f(a+h)$ approxi- mately. For small h; $\theta h (0<\theta<1)$ will further be small.

In view of this, we can neglect θh and write $f(a+h) = f(a) + h f'(a)$

approximately.

(ii) If $f(x)$ is continuous in $[a, a+h]$ and derivable in $(a, a+h)$ the value of $f(x)$ at the end point $a+h$ can be written in terms of $f(a)$, h and the derivative of $f(x)$ at some point in $(a, a+h)$.

Another interpretation of Lagrange's Mean value Theorem :—

Let a particular start at time $t=0$ at O and move along a straight line

Let it be at A at time $t=a$ and at B at time $t=b$ If P is any point

on OB. Let $OP = f(t)$.

Let the particle move continuously between $t=a$ and $t=b$ and the

velocity $f'(t)$ be defined at each t.

Then the particle attains the mean velocity $\frac{f(b)-f(a)}{b-a}$ at least once during the times $t=a$ and $t=b$ There exists a time c $(a<c<b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Theorem :- If f is derivable on (a,b) and

(i) $f'(x) \geq 0 \ \forall x \in (a,b)$, then f is increasing on (a,b).

(ii) $f'(x) \leq 0 \ \forall x \in (a,b)$ then f is decreasing on (a,b)

Note :- If $f'(x) > 0 \ \forall x \in (a,b)$ then f is strictly increasing on (a,b) and if $f'(x) < 0 \ \forall x \in (a,b)$ then f is strictly decreasing on (a,b)

11) Verify Lagrange's mean value theorem for $f(x) = x^3 - x^2 - 5x + 3$ in $[0, 4]$.

Sol:- Given that $f(x) = x^3 - x^2 - 5x + 3$ in $[0, 4]$.

we know that Every polynomial function is continuous on R.

The given polynomial function $f(x) = x^3 - x^2 - 5x + 3$ is continuous on $[0,4] \subset R$.

$$f(x) = x^3 - x^2 - 5x + 3$$

$$f'(x) = 3x^2 - 2x - 5$$

$f'(x)$ exists in $(0,4)$

$\therefore$ f is derivable in $(0, 4)$.

Hence by Lagrange's mean value theorem, there exists a point c in $(0,4)$ such that $f'(c) = \frac{f(4) - f(0)}{4 - 0}$

$$3c^2 - 2c - 5 = \frac{f(4) - f(0)}{4} \quad \text{— (1)}$$

$$f(4) = 4^3 - 4^2 - 5 \cdot 4 + 3 = 31$$

$$f(0) = 3$$

From (1), we have.

$$3c^2 - 2c - 5 = \frac{31 - 3}{4} = 7.$$

$$3c^2 - 2c - 12 = 0$$

$$c = \frac{2 \pm \sqrt{4 + 144}}{6} = \frac{1 \pm \sqrt{37}}{3}.$$

$$c = \frac{1 + \sqrt{37}}{3} \in (0, 4).$$

$\therefore$ Lagranges meanvalue theorem verified.

Calculate approximately $\sqrt[5]{245}$ by using Lagranges Mean value" Theorem.

Sol:- Let $f(x) = \sqrt[5]{x} = x^{1/5}$.

and. $a = 243$ $b = 245$.

$f'(x) = \frac{1}{5} x^{-4/5}$ $f'(c) = \frac{1}{5} c^{-4/5}$.

$\therefore$ By Lagranges mean value theorem, we have

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{f(245)-f(243)}{245-243} = \frac{1}{5} c^{-4/5}$$

$$f(245) - f(243) = \frac{2}{5} c^{-4/5}.$$

$$f(245) = f(243) + \frac{2}{5} c^{-4/5} \quad \text{—} (1)$$

c is lies between 243 and 245.

Take $c = 243$.

Then (1) be comes.

$$\sqrt[5]{245} = \sqrt[5]{243} + \frac{2}{5}(243)^{-\frac{4}{5}}$$

$$\sqrt[5]{245} = (3^5)^{1/5} + \frac{2}{5}(3^5)^{\frac{-4}{5}}$$

$$= 3 + \frac{2}{5} \cdot \frac{1}{81} = 3 + \frac{2}{405}$$

$$\sqrt[5]{245} = 3.0049.$$

If $a<b$, prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ using Lagranges Mean value theorem.

Deduce the following

(i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

(ii) $\frac{5\pi + 4}{20} < \tan^{-1} 2 < \frac{\pi + 2}{4}$.

sol:- Let $f(x) = \tan^{-1}x$ in $[a, b]$ for $0<a<b<1$.

$f(x) = \tan^{-1}x$ is continuous in $[a,b]$.

$f'(x) = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

f is derivable in (a, b)

$f(x) = \tan^{-1}x$ satisfies all the conditions of Lagrange's Mean value theorem

Hence by Lagrange's Mean value theorem.

There exists a point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Here $f'(x) = \frac{1}{1+x^2}$

$f'(c) = \frac{1}{1+c^2}$

Thus there exists a point c, $a<c<b$ such that

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a} \quad \text{——} (1)$$

We have $c \in (a,b)$ i.e $a<c<b$.

$a^2 < c^2 < b^2$

$1+a^2 < 1+c^2 < 1+b^2$

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2} \quad \text{——} (2)$$

From (1) and (2), We have

$$\frac{1}{1+b^2} < \frac{Tan^{-1}(b) - Tan^{-1}(a)}{b-a} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < Tan^{-1}(b) - Tan^{-1}(a) < \frac{b-a}{1+a^2}$$

<u>Deductions</u> :—

We have $\frac{b-a}{1+b^2} < Tan^{-1}(b) - Tan^{-1}(a) < \frac{b-a}{1+a^2}$ —— (3).

(i) Taking $a=1$ $b=\frac{4}{3}$ in (3), we get

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < Tan^{-1}\left(\frac{4}{3}\right) - Tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{\frac{4-3}{3}}{\frac{9+16}{9}} < Tan^{-1}\left(\frac{4}{3}\right) - Tan^{-1}\left(Tan\frac{\pi}{4}\right) < \frac{\frac{4-3}{3}}{2}$$

$$\frac{1}{3}\cdot\frac{9}{25} < Tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{6}$$

$$\frac{3}{25} + \frac{\pi}{4} < Tan^{-1}\left(\frac{4}{3}\right) < \frac{1}{6} + \frac{\pi}{4}.$$

(ii) Taking $a=1$ $b=2$ in (3), we get.

$$\frac{2-1}{1+4} < Tan^{-1}(2) - Tan^{-1}(1) < \frac{2-1}{1+1}$$

$$\frac{1}{5} < Tan^{-1}(2) - Tan^{-1}\left(Tan\frac{\pi}{4}\right) < \frac{1}{2}$$

$$\frac{1}{5} < Tan^{-1}(2) - \frac{\pi}{4} < \frac{1}{2}$$

$$\frac{\pi}{4}+\frac{1}{5}<\tan^{-1}(2)<\frac{\pi}{4}+\frac{1}{2}$$

$$\frac{5\pi+4}{20}<\tan^{-1}(2)<\frac{2\pi+2}{4}.$$

Cauchy's Mean Value Theorem :—

If $f:[a,b] \longrightarrow R$, $g:[a,b] \longrightarrow R$ are such that

(i) f, g are continuous on $[a,b]$.

(ii) f, g are differentiable on (a,b) and

(iii) $g'(x) \neq 0 \;\; \forall x \in (a,b)$.

then there exists a point $c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

Note:- We can derive Lagrange's Mean value Theorem from Cauchy's mean value Theorem by taking $g(x) = x$.

1) Find c of cauchy's Mean value theorem for $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$. in $[a, b]$ where $0 < a < b$.

Sol:- Given that $f(\sqrt{x}) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ defined on $[a,b]$.

The given functions $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ are continuous on $[a,b] \subset R^+$

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad g'(x) = \frac{-1}{2x\sqrt{x}} \text{ exists on } (a,b).$$

$\therefore$ f, g are derivable on $(a,b) \subseteq R^+$

Also $g'(x) \neq 0 \;\; \forall x \in (a,b) \subseteq R^+$

$\therefore$ All the conditions of cauchy's Mean value theorem are satisfied on (a,b).

$\therefore$ There exists $c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}} = -c.$$

$$-\sqrt{ab} = -c.$$

$$c = \sqrt{ab}.$$

Since $a>0,\ b>0$ $\sqrt{ab}$ is their geometric mean and.

we have $a < \sqrt{ab} < b$.

$\therefore$ $c \in (a,b)$ Which verifies Cauchy's mean value theorem.

(2) Find c of cauchy's Mean Value Theorem on $[a,b]$ for $f(x) = e^x$.

and $g(x) = e^{-x}$ $(a,b>0)$.

Sol:- Let $f(x) = e^x$ and $g(x) = e^{-x}$ on $[a,b] \subseteq R^+$.

We know that $f(x) = e^x$ and $g(x) = e^{-x}$ are continuous on $[a,b] \subseteq R^+$.

$f'(x) = e^x$ $\quad g'(x) = -e^{-x}$ exists on (a,b).

f and g are derivable on (a,b)

$g'(x) = -e^{-x} \neq 0 \quad \forall x \in (a,b)$.

$\therefore$ All the conditions of cauchy's Mean value theorem are satisfied.

By cachy's Mean value theorem.

There exists $c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}.$$

$$\frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = \frac{e^c}{\frac{-1}{e^c}}.$$

$$\frac{e^b - e^a}{\frac{e^a - e^b}{e^{a+b}}} = -e^{2c}.$$

$$-e^{a+b} = -e^{2c}$$

$$a+b = 2c$$

$$c = \frac{a+b}{2}.$$

$\therefore$ $c \in (a,b)$.

$\therefore$ Cauchy's Mean Value Theorem verified.

(3) Verify Cauchy's mean value theorem for the functions $f(x)$ and $f'(x)$. in $(1,e)$ given $f(x) = \log_e x$

Sol:- Given $f(x) = \log x$, $x \in (1,e)$

$$f'(x) = \frac{1}{x}$$

Let $g(x) = f'(x) = \frac{1}{x}$.

The functions $f(x)$ and $g(x)$ are continuous on $(1,e)$.

The functions $f(x)$ and $g(x)$ are derivable on $(1,e)$

$$g'(x) = \frac{-1}{x^2} \neq 0 \quad \forall x \in (1,e).$$

$\therefore$ All the conditions of cauchy's Mean Value theorem are satisfied.

$\therefore$ By cauchy's Mean Value Theorem.

There exists $c \in (1,e)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(e)-f(1)}{g(e)-g(1)}$

$$\frac{\frac{1}{c}}{\frac{-1}{c^2}} = \frac{\log e - \log 1}{\frac{1}{e}-1} = \frac{1-0}{\frac{1-e}{e}}$$

$$-c = \frac{-e}{e-1}$$

$$c = \frac{e}{e-1} \in (1,e)$$

$\therefore$ Cauchy's Mean value theorem verified.

(4) If $f(x) = \log x$ and $g(x) = x^2$ in $[a,b]$ with $b>a>1$ using Cauchy's Mean Value theorem P.T $\frac{\log b - \log a}{b-a} = \frac{a+b}{2c^2}$.

sol:- Given that $f(x) = \log x$ $g(x) = x^2$ in $[a,b]$.

The functions f and g are continuous on $[a,b]$.

The functions f and g are derivable on (a,b)

$$g'(x) = 2x \neq 0 \quad \forall x \in (a,b).$$

$\therefore$ All the conditions of Cauchy's Mean value theorem are satisfied.

$\therefore$ By Cauchy's Mean value theorem

There exists $c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

$$\frac{\log b - \log a}{b^2 - a^2} = \frac{\frac{1}{c}}{2c}.$$

$$\frac{\log b - \log a}{(b-a)(b+a)} = \frac{1}{2c^2}$$

$$\frac{\log b - \log a}{b-a} = \frac{b+a}{2c^2}.$$

(5) Discuss the applicability of Cauchy's Mean value theorem for the functions $f(x) = \frac{1}{x^2}$ $g(x) = \frac{1}{x}$ on $[a,b]$. $a>0$ $b>0$.

sol:- Given that $f(x) = \frac{1}{x^2}$ $g(x) = \frac{1}{x}$.

The functions f, g are continuous on $[a,b]$.

The functions f, g are derivable on (a,b)

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$f(x) = \frac{1}{x^2} \quad g(x) = \frac{1}{x}$

Diff. w.r.t "x", we get

$f'(x) = -\frac{2}{x^3} \quad g'(x) = -\frac{1}{x^2} \neq 0 \quad \forall x \in (a,b)$

$\therefore$ All the conditions of cauchy's mean value Theorem are satisfied

$\therefore$ By cauchy's Mean value theorem

There exists a point $c \in (a,b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

$$\frac{\frac{-2}{c^3}}{\frac{-1}{c^2}} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$\frac{2}{c} = \frac{a^2 - b^2}{a^2 b^2} \quad \frac{ab}{a-b}.$$

$$\frac{2}{c} = \frac{a+b}{ab}$$

$$c = \frac{2ab}{a+b} \in (a,b).$$

$\therefore$ Cauchy's Mean value Theorem verified.

Verify Cauchy's Mean value Theorem for $f(x) = \sin x$, $g(x) = \cos x$ on $[0, \frac{\pi}{2}]$.

sol:- Given that $f(x) = \sin x \quad g(x) = \cos x$ on $[0, \frac{\pi}{2}]$

i) We know that the functions $\sin x$ and $\cos x$ are continuous on $\mathbb{R}$

$\therefore f(x) = \sin x \quad g(x) = \cos x$ are continuous on $[0, \frac{\pi}{2}]$ $(\because [0, \frac{\pi}{2}] \subseteq \mathbb{R})$

(ii) $f(x) = \sin x \quad g(x) = \cos x$

Diff w.r.t "x", we get

$f'(x) = \cos x \quad g'(x) = -\sin x$.

$f'(x)$ and $g'(x)$ exists $\forall x \in (0, \frac{\pi}{2})$.

$\therefore$ f and g are derivable on $(0, \frac{\pi}{2})$.

(iii) $g'(x) = -\sin x \neq 0 \quad \forall x \in (0, \frac{\pi}{2})$.

All the conditions of cauchy's Mean value Theorem are satisfied.

$\therefore$ By Cauchy's Mean value Theorem

There exists a point $c \in (0, \frac{\pi}{2})$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(\frac{\pi}{2}) - f(0)}{g(\frac{\pi}{2}) - g(0)}$

$$\text{i.e} \quad \frac{\cos c}{-\sin c} = \frac{\sin\frac{\pi}{2} - \sin 0}{\cos\frac{\pi}{2} - \cos 0}$$

$$-\cot c = -1$$

$$\cot c = 1$$

$$c = \frac{\pi}{4} \in (0, \frac{\pi}{2})$$

$\therefore$ Cauchy's Mean value theorem is verified.

Generalised Mean Value Theorems :—

Taylor's Theorem :—

If $f:[a,b] \longrightarrow R$ is such that

(a) $f(x), f'(x), f''(x) \ldots f^{(n-1)}(x)$ is continuous on $[a,b]$.

(b) $f(x), f'(x), f''(x) \ldots f^{(n-1)}(x)$ exists on (a,b) (or) $f^{(n)}(x)$ exists on (a,b) and $p \in z^+$ then there exists a point $c \in (a,b)$ such that

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n .$$

where $R_n = \dfrac{(b-a)^p (b-c)^{n-p} f^{(n)}(c)}{(n-1)!\, p}$.

Note:- (1) schlomilch Roche's forms of remainder

$$R_n = \frac{(b-a)^p (b-c)^{n-p} f^{(n)}(c)}{(n-1)!\, p}$$

(2) Lagranges form of remainder

Putting $p=n$, we get

$$R_n = \frac{(b-a)^n f^{(n)}(c)}{n!}$$

(3) Cauchy's form of remainder

Putting $p=1$, we get

$$R_n = \frac{(b-a)(b-c)^{n-1} f^{(n)}(c)}{(n-1)!}$$

Another form of Taylor's Theorem :—

If $f:[a, a+h] \longrightarrow R$ is such that

(i) $f, f'(x), f''(x) \ldots f^{(n-1)}(x)$ is continuous on $[a, a+h]$.

(ii) $f^{(n)}(x)$ exists on $(a, a+h)$ and $p \in z^+$ then there exists a real number $0<\theta<1$ such that

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots + \frac{h^n}{(n-1)!} f^{(n-1)}(a) + R_n$$

where $$R_n = \frac{h^n (1-\theta)^{n-p} f^{(n)}(a+\theta h)}{p\,(n-1)!}$$

Note:- (1) Schlomilch Roches form of remainder

$$R_n = \frac{h^n (1-\theta)^{n-p} f^{(n)}(a+\theta h)}{p\,(n-1)!}$$

(2) Lagranges form of remainder, Putting $p=n$, we get

$$R_n = \frac{h^n f^{(n)}(a+\theta h)}{n!}$$

(3) cauchy's form of remainder, putting $p=1$, we get

$$R_n = \frac{h^n (1-\theta)^{n-1} f^{(n)}(a+\theta h)}{(n-1)!}$$

Note:- (1) Taylor's theorem play an important role in differentiation. The values of a function and its successive derivatives at a point help us in finding the value of the function in the neighbour-hood of that point using Taylor's theorem. That is, Taylor's theorem provides expansion of $f(a+h)$ in ascending powers of h and the derivatives of f at a.

(2) Let $f:[a,b] \longrightarrow R$ is such that (a) $f^{(n-1)}$ is continuous on $[a,b]$.
b) $f^{(n-1)}$ is derivable on (a,b) and $p \in Z^+$ Then for each $x \in (a,b)$
$f^{(n-1)}$ is continuous on $[a,x]$ and derivable on (a,x)

$\therefore$ By Taylors theorem, There exists $c \in (a,x)$ such that

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^p (x-c)^{n-p} f^{(n)}(c)}{(n-1)!\,p}$$

is called Taylor series expansion of $f(x)$ about $x=a$, suppose the remainder after nth term tends to 0 as $n \longrightarrow \infty$.

Maclaurin's Theorem :—

If $f:[0,x] \longrightarrow R$ is such that

(i) $f^{(n-1)}$ is continuous on $[0,x]$

(ii) $f^{(n-1)}$ is derivable on $(0,x)$ and $p \in Z^+$ then there exists a real number $\theta \in (0,1)$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n (1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(\theta x).$$

Note:- (1) Schlomich Roche's form of remainder

$$R_n = \frac{x^n (1-\theta)^{n-p} f^{(n)}(\theta x)}{p(n-1)!}$$

(2) Lagrange's form of remainder

Putting $p=n$, we get $R_n = \dfrac{x^n f^{(n)}(\theta x)}{n!}$

(3) Cauchy's form of remainder.

Putting $p=1$, we get $R_n = \dfrac{x^n (1-\theta)^{n-1} f^{(n)}(\theta x)}{(n-1)!}$

(1) Obtain the Maclaurin's series expansion of the functions

1(a) e^x 1(b) $\sin x$ (c) $\sinh x$.

Sol:- 1(a) Let $f(x) = e^x$

The Maclaurin's series expansion of the fun. f is given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad (1)$$

$f(x) = e^x$ At $x=0$ $f(0) = e^0 = 1$

$f'(x) = e^x$ At $x=0$ $f'(0) = e^0 = 1$

$f''(x) = e^x$ At $x=0$ $f''(0) = e^0 = 1$

$f'''(x) = e^x$ At $x=0$ $f'''(0) = e^0 = 1$

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Sub. all the above values in (1), we get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

1(b) Let $f(x) = \sin x$.

The Maclaurin's series expansion of the fun. f is given by

$$f(0) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad (1)$$

$f(x) = \sin x$ At $x=0$ $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ At $x=0$ $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ At $x=0$ $f''(0) = -\sin 0 = 0$.

$f'''(x) = -\cos x$ At $x=0$ $f'''(0) = -\cos 0 = -1$

sub. all the above values in (1), we get.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

1(c) Let $f(x) = \sinh x$

$$f(x) = \frac{e^x + e^{-x}}{2}$$

The Maclaurin's series of the fun. f is given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad (1)$$

$$f(x) = \frac{e^x - e^{-x}}{2} \qquad \text{At } x=0, \; f(0) = 0$$

$$f'(x) = \frac{e^x + e^{-x}}{2} \qquad \text{At } x=0, \; f'(0) = 1$$

$$f''(x) = \frac{e^x - e^{-x}}{2} \qquad \text{At } x=0, \; f''(0) = 0$$

$$f'''(x) = \frac{e^x + e^{-x}}{2} \qquad \text{At } x=0, \; f'''(0) = 1$$

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Sub. above values in (1), we get

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

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Note :- The Taylor series expansion of the function f about the point $x=a$ is given by 20

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

(1) obtain the Taylor's series expansion of the function e^x about $x=-1$

(OR) obtain the Taylor's series expansion of $f(x) = e^x$ in powers of $x+1$.

Sol: Let $f(x) = e^x$.

Put $x+1 = t$

$x = t-1$

$f(x) = e^x = e^{t-1}$.

$$f(x) = e^{-1} e^t$$

$$= \frac{1}{e}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]$$

$$e^x = \frac{1}{e}\left[1 + (x+1) + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \cdots\right]$$

(OR)

Let $f(x) = e^x$.

The Taylor series expansion of the function $f(x)$ in powers of $x+a$ is given by

$$f(x) = f(a) + \frac{x+a}{1!} f'(a) + \frac{(x+a)^2}{2!} f''(a) + \frac{(x+a)^3}{3!} f'''(a) + \cdots$$

Here we have to find expansion of $f(x) = e^x$ in powers of $x+1$

Then $$f(x) = f(-1) + \frac{x+1}{1!} f'(-1) + \frac{(x+1)^2}{2!} f''(-1) + \frac{(x+1)^3}{3!} f'''(-1) + \cdots \quad (1)$$

$f(x) = e^x$ At $x=-1$, $f(-1) = e^{-1}$

$f'(x) = e^x$ At $x=-1$ $f'(-1) = e^{-1}$

$f''(x) = e^x$ At $x=-1$ $f''(-1) = e^{-1}$

sub. all these values in ①, we get

$$e^x = e^{-1} + (x+1)e^{-1} + \frac{(x+1)^2}{2!}e^{-1} + \frac{(x+1)^3}{3!}e^{-1} + \cdots$$

$$e^x = e^{-1}\left[1 + (x+1) + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \cdots\right]$$

Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$ and hence deduce that $\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$

Sol:- Let $f(x) = \log(1+e^x)$.

The Maclaurin's series of the function is given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad \text{—— (1)}$$

$f(x) = \log(1+e^x)$ At $x=0$, $f(0) = \log(1+e^0) = \log 2$

$f'(x) = \frac{e^x}{1+e^x}$ At $x=0$, $f'(0) = \frac{1}{2}$

$$f''(x) = \frac{(1+e^x)e^x - e^x . e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

At $x=0$, $f''(0) = \frac{1}{4}$

$$f'''(x) = \frac{(1+e^x)^2 e^x - e^x\, 2(1+e^x) . e^x}{(1+e^x)^4} = \frac{(1+e^x)\left[e^x + e^{2x} - 2e^{2x}\right]}{(1+e^x)^4}$$

$$f'''(x) = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

At $x=0$, $f'''(0) = \frac{1-1}{(1+e^0)^3} = 0$

$$f^{iv}(x) = \frac{(1+e^x)^3 (e^x - 2e^{2x}) - (e^x - e^{2x})\, 3(1+e^x)^2 e^x}{(1+e^x)^6}$$

$$f^{iv}(x) = \frac{(1+e^x)(e^x - 2e^{2x}) - (e^x - e^{2x}) 3e^x}{(1+e^x)^4}$$

At $x=0$, $f^{iv}(0) = \frac{(1+1)(1-2) - (1-1)3}{(1+1)^4} = \frac{-1}{8}$.

Sub. all these values in ①, we get

$$\log(1+e^x) = \log 2 + x \cdot \frac{1}{2} + \frac{x^2}{2!} \cdot \frac{1}{4} + \frac{x^3}{3!}(0) + \frac{x^4}{4}\left(-\frac{1}{8}\right) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots \quad \text{——— ②}$$

Deduction :-

Diff ② w.r.t x, we get.

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

→ Show that $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\dfrac{x^3}{3!} + \cdots$

Sol: Let $f(x) = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

The Maclaurin's series of the function f is given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$f(x) = \frac{\sin^{-1}x}{\sqrt{1-x^2}}, \quad \text{At } x=0, \; f(0) = 0.$$

$$\sqrt{1-x^2}\, f(x) = \sin^{-1}x \quad \text{——— ①}$$

Diff ① w.r.t "x", we get

$$\sqrt{1-x^2}\, f'(x) + f(x) \frac{-2x}{2\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$(1-x^2) f'(x) - x f(x) = 1 \quad \text{——— ②}$$

At $x=0$, $f'(0) = 1$ (∵ from ②)

Diff ② w.r.t 'x', we get

$$(1-x^2) f''(x) - 2x f'(x) - f(x) - \cancel{2xf'(x)} = 0.$$

$(1-x^2) f''(x) - 3x f'(x) - f(x) = 0$ — 3

At $x=0$, $f''(0) = f(0) = 0$.

Diff 3 w.r.t x, we get

$(1-x^2) f'''(x) - 2x f''(x) - 3f'(x) - 3x f''(x) - f'(x) = 0$

$(1-x^2) f'''(x) - 5x f''(x) - 4f'(x) = 0$ — 4

At $x=0$, $f'''(0) = 4f'(0)$

$f'''(0) = 4$.

Sub. all these values in maclaurin's series, we get

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + \cdots$$

→ Using Taylor's series obtain the value of sin32° correct to four decimal places.

Sol:- Let $f(x) = \sin x$ in $[30°, 32°]$.

We know that the Taylor's series.

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \cdots$$

Here $a = 30°$ $b = 32°$

$$b-a = 32° - 30° = 2° = 2 \times \frac{\pi}{180} = 0.0349$$

$f(x) = \sin x$ $\qquad f(a) = f(30°) = \sin 30° = \frac{1}{2}$.

$f'(x) = \cos x$ $\qquad f'(a) = f'(30°) = \cos 30° = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x$ $\qquad f''(a) = f''(30°) = -\sin 30° = -\frac{1}{2}$.

$f'''(x) = -\cos x$ $\qquad f'''(a) = f'''(30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$.

. . . .

Sub. all these values in above taylors series, weget

$$\sin 32° = \frac{1}{2} + \frac{0.0349}{1!}\left(\frac{\sqrt{3}}{2}\right) + \frac{(0.0349)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(0.0349)^3}{3!}\left(-\frac{\sqrt{3}}{2}\right).$$

$$= 0.5 + 0.03023 - 0.0003045 - 0.0000061356.$$

$\sin 32° = 0.5299.$

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Functions of Several Variable. {Module-3}

Partial differentiations—

- Let $z = f(x,y)$ be a function of two variables x & y.

then $\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$ exist is said to be partial derivative or partial differential coefficient of z (or) $f(x,y)$ w.r.t x.

It is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x

The partial derivative of $z = f(x,y)$ w.r.t 'x', keeping y as constant.

Similarly, the partial derivative of $z = f(x,y)$ w.r.t y keeping x as constant and is defined as

$$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$ and is denoted by

$\frac{\partial z}{\partial y}$ (or) $\frac{\partial f}{\partial y}$ (or) f_y

Higher Order partial derivatives:

In general the first order partial derivatives $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are also functions of x & y. and they can be differentiated repeatedly to get higher order partial derivatives.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

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$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Q) Find the first and Second Order partial derivatives of $f = ax^2 + 2hxy + by^2$ and Verify $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Sol: G.T, $f = ax^2 + 2hxy + by^2$

diff w.r.t 'x' partially, we get

$$f_x = \frac{\partial f}{\partial x} = 2ax + 2hy$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2a$$

diff w.r.t 'y' partially, we get.

$$f_y = \frac{\partial f}{\partial y} = 2hx + 2by$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2b$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2hx + 2hy) = 2h$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2ax + 2hy) = 2h$$

$$f_{xy} = f_{yx}.$$

~~Q) If~~ $\cancel{r^2 = x^2+y^2+z^2}$ and $\cancel{f = r^n}$. P.T $\cancel{f_{xx}+f_{yy}+f_{zz} = n(n+1)r^{n-2}}$

Sol:

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Q) Find 1^{st} & 2^{nd} Order partial derivatives of $f = x^3+y^3-3axy$ and verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \cdot \partial x}$

Sol: G.T, $f = x^3+y^3-3axy$

diff w.r.t 'x' partially, we get

$$f_x = \frac{\partial f}{\partial x} = 3x^2 - \cancel{3y^2 \frac{\partial y}{\partial x}} - 3ay = 3x^2 - 3ay.$$

$$f_{xx} = 6x$$

diff w.r.t 'y' partially, we get

$$f_y = 3y^2 - 3ax$$

$$f_{yy} = 6y.$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3y^2-3ax) = -3a$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \cdot \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2-3ay) = -3a$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Q) Verify that $f_{xy} = f_{yx}$ for the function $f = \tan^{-1}\left(\frac{x}{y}\right)$

Sol: G.T, $f = \tan^{-1}\left(\frac{x}{y}\right)$

diff w.r.t x, we get.

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right)$$

$$f_x = \frac{\partial f}{\partial x} = \frac{y}{y^2+x^2}$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{y}{y^2+x^2}\right)$$

$$= \frac{(y^2+x^2)1 - y(2y)}{}$$

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$$f_{yx} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$f = \tan^{-1}\left(\frac{x}{y}\right)$$

diff w.r.t 'y' partially, we get.

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right)$$

$$f_y = \frac{\partial f}{\partial y} = \frac{-x}{x^2+y^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(-1) - (-x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$f_{xy} = f_{yx}$$

Q) If $x = \frac{1}{\sqrt{x^2+y^2+z^2}}$. P.T $f_{xx}+f_{yy}+f_{zz}=0$

Sol: G.T, $f = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$$f = (x^2+y^2+z^2)^{-1/2}$$

diff w.r.t x partially

$$f_x = \frac{\partial f}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$$

$$f_x = -x(x^2+y^2+z^2)^{-3/2}$$

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diff w.r.t 'x', partially, we get

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = -\left[1(x^2+y^2+z^2)^{-3/2} + \left(\frac{-3x}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x)\right]$$

$$f_{xx} = -(x^2+y^2+z^2)^{-3/2}\left[1 - 3x^2(x^2+y^2+z^2)^{-1}\right]$$

Similarly,

$$f_{yy} = -(x^2+y^2+z^2)^{-3/2}\left[1-3y^2(x^2+y+z^2)^{-1}\right]$$

$$f_{zz} = -(x^2+y^2+z^2)^{-3/2}\left[1-3z^2(x^2+y^2+z^2)^{-1}\right]$$

$$f_{xx}+f_{yy}+f_{zz} = -(x^2+y^2+z^2)^{-3/2}\left[3-(x^2+y^2+z^2)^{-1}3(x^2+y^2+z^2)\right]$$

$$f_{xx}+f_{yy}+f_{zz} = 0.$$

Q) If $f = \log(x^2+y^2+z^2)$. P.T $(x^2+y^2+z^2)(f_{xx}+f_{yy}+f_{zz}) = 2$

Sol: Given, $f = \log(x^2+y^2+z^2)$

diff w.r.t 'x' partially, we get.

$$f_x = \frac{1}{x^2+y^2+z^2}(2x)$$

$$f_{xx} = \frac{2(x^2+y^2+z^2)-2x(2x)}{(x^2+y^2+z^2)^2} = \frac{2(x^2+y^2+z^2)-4x^2}{(x^2+y^2+z^2)^2}$$

$$f_{xx} = \frac{2y^2+2z^2-2x^2}{(x^2+y^2+z^2)^2}$$

diff f w.r.t 'y' partially, we get.

$$f_y = \frac{1}{x^2+y^2+z^2}(2y)$$

$$f_{yy} = \frac{2(x^2+y^2+z^2)-2y(2y)}{(x^2+y^2+z^2)^2}$$

$$f_{yy} = \frac{2x^2-2y^2+2z^2}{(x^2+y^2+z^2)^2}$$

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diff 'f' w.r.t 'z' partially, we get

$$f_z = \frac{1}{x^2+y^2+z^2}(2z)$$

$$f_{zz} = \frac{2(x^2+y^2+z^2) - 2z(2z)}{(x^2+y^2+z^2)^2}$$

$$f_{zz} = \frac{2x^2+2y^2-2z^2}{(x^2+y^2+z^2)^2}$$

$$\therefore f_{xx}+f_{yy}+f_{zz} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} = \frac{2}{x^2+y^2+z^2}$$

$$\Rightarrow (x^2+y^2+z^2)(f_{xx}+f_{yy}+f_{zz}) = \cancel{(x^2+y^2+z^2)}\frac{2}{\cancel{x^2+y^2+z^2}}$$

$$= 2$$

Q) Verify that $f_{xx}+f_{yy}=0$ if $f=\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$

Sol: G.T, $f = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$

diff w.r.t 'x', partially

$$f_x = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \cdot \frac{2y(x^2-y^2) - 2xy(2x)}{(x^2-y^2)^2}$$

$$= \frac{\cancel{(x^2-y^2)^2}}{(x^2-y^2)^2+(2xy)^2} \cdot \frac{2x^2y-2y^3-4x^2y}{\cancel{(x^2-y^2)^2}}$$

$$= \frac{-2x^2y-2y^3}{x^4+y^4-2x^2y^2+4x^2y^2} = \frac{-2y(x^2+y^2)}{x^4+y^4+2x^2y^2} = \frac{-2y\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{2}}}$$

$\cancel{f_{xx}}$ $f_x = \frac{-2y}{x^2+y^2}$

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$$f_{xx} = \frac{4xy}{(x^2+y^2)^2}$$

diff w.r.t 'y' partially,

$$f_y = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \frac{2x(x^2-y^2) - 2xy(-2y)}{(x^2-y^2)^2}$$

$$= \frac{\cancel{(x^2-y^2)^2}}{x^4+y^4-2x^2y^2+4x^2y^2} \cdot \frac{2x^3-2xy^2+4xy^2}{\cancel{(x^2-y^2)^2}}$$

$$= \frac{2x^3+2xy^2}{x^4+y^4+2x^2y^2} = \frac{2x(x^2+y^2)}{(x^2+y^2)^{\cancel{2}}} = \frac{2x}{x^2+y^2}$$

$$\Rightarrow \cancel{f_{xx}+f_{yy}} \quad f_{yy} = \frac{-4xy}{(x^2+y^2)^2}$$

$$\Rightarrow f_{xx}+f_{yy} = \frac{4xy}{(x^2+y^2)^2} - \frac{4xy}{(x^2+y^2)^2}$$

$$= 0$$

$$\therefore \quad f_{xx}+f_{yy} = 0$$

Q) If $r^2 = x^2+y^2+z^2$ and $f = r^n$, P.T $f_{xx}+f_{yy}+f_{zz} = n(n+1)r^{n-2}$.

Sol: G.T, $r^2 = x^2+y^2+z^2$

$f = r^n$

$\cancel{f} = (x^2+y^2+z^2)^{n/2}$

diff partially with 'x'

$$\frac{\partial f}{\partial x} = \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x)$$

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$$f_x = \frac{\partial f}{\partial x} = nx(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = n\left[1(x^2+y^2+z^2)^{\frac{n-2}{2}} + x\left(\frac{n-2}{2}\right)(x^2+y^2+z^2)^{\frac{n-2}{2}}\right].$$

$$f_{xx} = n\left[(x^2+y^2+z^2)^{\frac{n-2}{2}} + (n-2)x^2(x^2+y^2+z^2)^{\frac{n-4}{2}}\right]$$

$$f_{xx} = n\left[r^{n-2} + (n-2)x^2 r^{n-4}\right] \qquad [\because x^2+y^2+z^2 = r^2]$$

$||^{ly}$ $$f_{yy} = n\left[r^{n-2} + (n-2)y^2 r^{n-4}\right]$$

$$f_{zz} = n\left[r^{n-2} + (n-2)z^2 r^{n-4}\right]$$

$$f_{xx}+f_{yy}+f_{zz} = n\left[3r^{n-2} + (n-2)r^{n-4}(x^2+y^2+z^2)\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-4}r^2\right]$$

$$= nr^{n-2}\left[3+(n-2)\right]$$

$$= n(n+1)r^{n-2}.$$

Q) If $f = e^{xyz}$ S.T. $f_{xyz} = (1+3xyz+x^2y^2z^2)e^{xyz}$.

Sol: G.T, $f = e^{xyz}$. —— ①

diff w.r.t 'x' partially

$$f_x = e^{xyz}(yz)$$

$$f_x = yze^{xyz}$$

diff w.r.t 'y' partially

$$f_y = e^{xyz}(xz)$$

$$f_y = xze^{xyz}.$$

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diff ① w.r.t 'z' partially.

$$\frac{\partial f}{\partial z} = f_z = e^{xyz} \cdot (xy)$$

$$f_z = xy\, e^{xyz} \quad \text{—— ②}$$

diff ② w.r.t 'y'. partially.

$$\frac{\partial^2 f}{\partial y\, \partial z} = \frac{\partial}{\partial y}\left(xy\, e^{xyz}\right) = x \frac{\partial}{\partial y}\left(y\, e^{xyz}\right)$$

$$= x\left[e^{xyz} + xyz\, e^{xyz}\right] = (x + x^2 yz)\, e^{xyz} \quad \text{—— ③}$$

diff ③ w.r.t 'x' partially

$$\frac{\partial^3 f}{\partial x\, \partial y\, \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y\, \partial z}\right]$$

$$= \frac{\partial}{\partial x}\left[(x + x^2 yz)\, e^{xyz}\right]$$

$$= (1 + 2xyz)\, e^{xyz} + (x + x^2 yz)\, e^{xyz}(yz)$$

$$= (1 + 2xyz + xyz + x^2 y^2 z^2)\, e^{xyz}$$

$$= (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}$$

Hence $\dfrac{\partial^3 f}{\partial x\, \partial y\, \partial z} = f_{xyz}$

$$= (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}$$

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Q) If $z = f(x+ay) + g(x-ay)$. P/T, $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

Sol: G.T,

$z = f(x+ay) + g(x-ay)$ — ①

diff ① w.r.t to 'x' partially,

$$\frac{\partial z}{\partial x} = f'(x+ay)1 + g'(x-ay)1$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay)1 + g''(x-ay)1$$

diff ① w.r.t 'y' partially, we get

$$\frac{\partial z}{\partial y} = f'(x+ay)a + g'(x-ay)(-a)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+ay)a^2 + g''(x-ay)(-a)^2$$

$$\frac{\partial^2 z}{\partial y^2} = a^2\left[f''(x+ay) + g''(x-ay)\right]$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Q) If $z = \log(e^x + e^y)$. S.T, $rt - s^2 = 0$

where $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$

Sol: G.T, $z = \log(e^x + e^y)$

diff w.r.t 'x' partially

$$\frac{\partial z}{\partial x} = \frac{1}{e^x + e^y}(e^x)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{e^x(e^x+e^y) - e^x(e^x)}{(e^x+e^y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\cancel{e^x} + e^y e^x}{e^x + e^y} = \frac{\cancel{e^x}(1+e^y)}{\cancel{e^x+e^y}} = \frac{e^{x+y}}{(e^x+e^y)^2}$$

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diff w.r.t 'y' partially.

$$\frac{\partial z}{\partial y} = \frac{1}{e^x+e^y}(e^y)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{e^y(e^x+e^y) - e^y(e^y)}{(e^x+e^y)^2} = \frac{e^{xy}+\cancel{e^y}}{e^x+\cancel{e^y}} = \frac{e^{x+y}}{(e^x+e^y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{e^y\cancel{(1+e^x)}}{\cancel{e^x+e^y}} \quad \frac{e^{x+y}}{(e^x+e^y)^2}$$

$$\therefore r = \frac{\partial^2 z}{\partial x^2} = \frac{e^x\cancel{(1+e^y)}}{\cancel{e^x+e^y}} \quad \frac{e^{x+y}}{(e^x+e^y)^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{e^y\cancel{(1+e^x)}}{\cancel{e^x+e^y}} \cdot \frac{e^{x+y}}{(e^x+e^y)^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \cancel{\frac{\partial z}{\partial x}} \cdot \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{e^y}{e^x+e^y}\right) = e^y\left[\frac{-1}{(e^x+e^y)^2}\right]$$

$$\Rightarrow rt-s^2 = \frac{e^x(1+e^y)}{e^x+e^y} \cdot \frac{e^y(1+e^x)}{e^x+e^y} + \left[\frac{-e^y}{(e^x+e^y)^2}\right]^2$$

$$= \frac{e^x.e^y(1+e^x)(1+e^y)}{(e^x+e^y)^2} - \frac{e^{2y}}{(e^x+e^y)^2}$$

$$= \frac{e^{xy}\left[1+e^y+e^x+e^{xy}\right]}{(e^x+e^y)^2} - \frac{e^{2y}}{(e^x+e^y)^2}$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[\frac{e^y}{e^x+e^y}\right] = e^x e^y\left[\frac{-1}{(e^x+e^y)^2}\right]$$

$$\Rightarrow rt-s^2 = \frac{e^{x+y}}{(e^x+e^y)^2} \cdot \frac{e^{x+y}}{(e^x+e^y)^2} - \left(e^{x+y}\right)^2\left[\frac{1}{(e^x+e^y)^4}\right]$$

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$$= \frac{(e^{x+y})^2}{(e^x+e^y)^4} - \frac{(e^{x+y})^2}{(e^x+e^y)^4}$$

$$= 0$$

$\therefore \; rt - s^2 = 0.$

Q) If $x^x . y^y . z^z = e$. S.T, at $x=y=z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

Sol: G.T, $x^x . y^y . z^z = e$ —①

Taking logarithm b.s of ①, we get

$$\log_e (x^x y^y z^z) = \log_e e$$

$$\log_e x^x + \log_e y^y + \log_e z^z = 1$$

$$x \log_e x + y \log_e y + z \log_e z = 1$$

$$z \log_e z = 1 - x \log_e x - y \log_e y \quad —②$$

Diff ② w.r.t 'x', partially, we get.

$$\left(z . \frac{1}{z} \frac{\partial z}{\partial x} + \log z . \frac{\partial z}{\partial x}\right) = -\left(x . \frac{1}{x} + \log x . 1\right)$$

$$\frac{\partial z}{\partial x}(1+\log z) = -(1+\log x)$$

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \quad —③$$

Similarly, $\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{(1+\log z)}$ —④

At $x=y=z$, $\frac{\partial z}{\partial x} = -1$, $\frac{\partial z}{\partial y} = -1$ [$\because$ from ③ & ④]

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$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-(1+\log y)}{(1+\log z)}\right)$$

$$= -(1+\log y)\,\frac{\partial}{\partial x}\left[(1+\log z)^{-1}\right]$$

$$= -(1+\log y)(-1)(1+\log z)^{-2}\,\frac{1}{z}\cdot\frac{\partial z}{\partial x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{z}\,\frac{(1+\log y)}{(1+\log z)^2}\,\frac{\partial z}{\partial x}$$

$$= \frac{1}{z}\cdot\frac{1+\log y}{(1+\log z)^2}\,(-1)$$

At $x=y=z$,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x}\cdot\frac{1+\log x}{(1+\log x)^2}\,(-1)$$

$$= -\frac{1}{x}\cdot\frac{1}{(1+\log x)}$$

$$= -\frac{1}{x}\cdot\frac{1}{\log_e e+\log_e x}$$

$$= -x^{-1}\left(\log_e e+\log_e x\right)^{-1}$$

$$= -x^{-1}\left(\log_e ex\right)^{-1}$$

$$= -(x\log_e(ex))^{-1}$$

Q) If $z = \frac{x^2+y^2}{x+y}$. S.T, $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Sol: G.T, $z = \frac{x^2+y^2}{x+y}$

diff w.r.t x partially

$$\frac{\partial z}{\partial x} = \frac{2x(x+y)-(x^2+y^2)(1)}{(x+y)^2}$$

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$$\frac{\partial z}{\partial x} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{2x^2-(x-y)^2}{(x+y)^2}$$

diff 'z' w.r.t 'y' partially.

$$\frac{\partial z}{\partial y} = \frac{2y(x^2+y^2)-(x^2+y^2)}{(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} = \frac{2y^2-(x-y)^2}{(x+y)^2}.$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{x^2-y^2+2xy}{(x+y)^2} - \frac{y^2-x^2+2xy}{(x+y)^2}\right]^2$$

$$= \frac{4\left[(x^2-y^2)^2\right]}{(x+y)^4} = 4\left[\frac{(x+y)^2(x-y)^2}{(x+y)^4}\right]$$

$$4\left[1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right] = 4\left[1-\frac{(x^2-y^2+2xy)}{(x+y)^2} - \frac{(y^2-x^2+2xy)}{(x+y)^2}\right]$$

$$= 4\frac{\left[x^2+y^2+2xy-x^2+y^2-2xy-y^2+x^2-2xy\right]}{(x+y)^2}.$$

$$= 4\left[\frac{x^2+y^2-2xy}{(x+y)^2}\right]$$

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$$= \frac{4xy^2 + 4y^3 + 4x^2y + 4x^3}{((x+y)^2)^2} - \frac{2[x^2y^2 - x^4 + 2x^3y + 2xy^3 - 2x^3y + 4x^2y^2 - y^4 + x^2y^2 \ldots]}{[(x+y)^2]^2}$$

$$= \frac{4(x^3+y^3) + 4xy(x+y) + 2(x^4+y^4) - 2(6x^2y^2)}{[(x+y)^2]^2}$$

8) If $f = \log(x^3+y^3+z^3-3xyz)$, S.T. $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 = \frac{-1}{(x+y \ldots}$

Sol:- G.T, $f = \log(x^3+y^3+z^3-3xyz)$.

diff w.r.t 'x' partially.

$$\frac{\partial f}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial f}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial f}{\partial z} = \frac{3z^2 - 3xy}{(x^3+y^3+z^3-3xyz)}$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z}$$

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$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 f=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)f$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\frac{3}{x+y+z}$$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

Q). If $f=e^{x^y}$. $\cancel{\frac{\partial^2 f}{\partial y.\partial x}}$ find $\frac{\partial^2 f}{\partial y.\partial x}$:

Sol: G.T, $f=e^{x^y}$

diff w.r.t 'x', partially, we get.

$$\frac{\partial f}{\partial x}=e^{x^y}\; y x^{y-1}$$

$$=x^{-1}e^{x^y}y x^y.$$

diff w.r.t 'y' partially, we get

$$\frac{\partial^2 f}{\partial y.\partial x}=x^{-1}\left[e^{x^y}x^y+e^{x^y}y x^y\log x+y x^y e^{x^y}.x^y.\log x\right]$$

$$=x^{-1}e^{x^y}x^y\left(1+y\log x+y x^y\log x\right)$$

$$=e^{x^y}x^{y-1}\left(1+y\log x+y x^y\log x\right)$$

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Jacobian :

If u & v are functions of two independent variables x & y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \text{i.e} \quad \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$\frac{\partial(u,v)}{\partial(x,y)}$ (or) $J\left(\frac{u,v}{x,y}\right)$ is called the Jacobian of u,v w.r.t x & y. It is denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ (or) } J\left(\frac{u,v}{x,y}\right)$$

If u,v,w are functions of 3 independent variables x,y,z then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad \text{i.e} \quad \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

is Jacobian of u,v,w w.r.t x,y,z.
& is denoted by $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ (or) $J\left(\frac{u,v,w}{x,y,z}\right)$

Properties :-

i) $JJ' = 1$ i.e $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

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ii) If u, v are function of r, s & r, s are functions of all x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

Functionally dependence :

Let u & v be two functions of x & y suppose these functions connected by the relation $f(u,v)=0$ where f is differentiable the we say that u & v are functionally dependent.

We shall prove that the condition for functional dependence is $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

* **Theorem :**

If the functions u & v are independent variables x & y are functionally dependent then the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ vanish.

Note : If the Jacobian $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ then u & v are said to be functionally independent.

Q) $\boxed{\text{Type-1}}$

If $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$

Also show that $JJ' = 1$ i.e $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$.

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Sol: G.T, $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$.

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin r & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r.$$

$x^2 = r^2\cos^2\theta$, $y^2 = r^2\sin^2\theta$

$r^2 = x^2 + y^2$.

$r = \sqrt{x^2+y^2}$

$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$r = \sqrt{x^2+y^2}$

$$\frac{\partial r}{\partial x} = \frac{2 \cdot x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

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$$\frac{\partial r}{\partial y} = \frac{2y}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\theta = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{x^2+y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{1}{(x^2+y^2)^{1/2}} = \frac{1}{(r^2)^{1/2}} = \frac{1}{r}$$

$$\therefore JJ' = \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1$$

Q) P.T $JJ' = 1$ for $x = e^v \sec u$, $y = e^v \tan u$

Sol: G.T, $x = e^v \sec u$

$y = e^v \tan u$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\frac{\partial x}{\partial u} = e^v [\sec u \cdot \tan u]$$

$$\frac{\partial x}{\partial v} = \sec u e^v$$

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$$\frac{\partial y}{\partial u} = e^{u}(\sec^2 u)$$

$$\frac{\partial y}{\partial v} = e^{v} \tan u .$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} e^{v}|\sec u . \tan u| & e^{v}\sec u \\ e^{v}(\sec^2 u) & e^{v}\tan u \end{vmatrix}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)}$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)}$$

$$J = e^{v}\sec u \tan u \, e^{v}\tan u - e^{v}\sec u \, e^{v}\sec^2 u$$

$$= e^{2v}\sec^3 u\left[-\cos^2 u\right] = -e^{2v}\sec u .$$

$$J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$u = \operatorname{Cosec}^{-1}\left(\frac{x}{y}\right) , \quad v = \frac{1}{2}\log|x^2 - y^2|$$

$$\frac{\partial u}{\partial x} = \frac{-1}{\frac{x}{y}\sqrt{\frac{x^2-y^2}{y^2}}} \times \frac{1}{y} = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\frac{x}{y}\sqrt{\frac{x^2-y^2}{y^2}}} \times \frac{x}{y^2} = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2-y^2} \cdot \frac{1}{2} = \frac{x}{x^2-y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{2}\left(\frac{-2y}{x^2-y^2}\right) = \frac{-y}{x^2-y^2}$$

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$$J' = \begin{vmatrix} \frac{-y}{x\sqrt{x^2-y^2}} & \frac{x}{x^2-y^2} \\ \frac{1}{\sqrt{x^2-y^2}} & \frac{-y}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$= \frac{y^2}{x(x^2-y^2)^{1/2}} - \frac{x}{(x^2-y^2)^{3/2}} = \frac{y^2-x^2}{x(x^2-y^2)^3}$$

$$= \frac{-1}{x\sqrt{x^2-y^2}} = \frac{-1}{e^v \sec u\,(e^v)}$$

$$JJ' = \frac{-1}{e^{2v}\sec u} \times e^{-2v}\sec u = 1$$

Q) P.T $JJ' = 1$ for $x = \underline{u(1-v)}$, $y = uv$.

Sol: G.T, $x = u - uv$

$y = uv.$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = u - uv$ — $y = uv$.

$\frac{\partial x}{\partial u} = 1 - v$, $\frac{\partial x}{\partial v} = -u$ $\frac{\partial y}{\partial u} = v$, $\frac{\partial y}{\partial v} = u$.

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$$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= u - uv + uv.$

$J = u$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$x = u - uv \ , \ y = uv$

$u = x + y$

$\frac{x}{y} = \frac{u - uv}{uv}$

$\frac{x}{y} = \frac{1}{v} - 1 \Rightarrow \frac{1}{v} = 1 + \frac{x}{y} = \frac{y+x}{y}$

$V = \frac{y}{x+y}$

$u = x + y$

$\frac{\partial u}{\partial x} = 1 \ , \ \frac{\partial u}{\partial y} = 1$

$V = \frac{y}{x+y}$

$\frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2}$

$\frac{\partial v}{\partial y} = \frac{(x+y)1 - y(1)}{(x+y)^2} = \frac{x}{(x+y)^2}$

$$J' = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

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$$= \frac{x+4y}{(x+y)^2} = \frac{1}{x+y}$$

$$J' = \frac{1}{4}$$

$$JJ' = 4\frac{1}{4} = 1$$

Q) P.T $JJ' = 1$, for $x = uv$, $y = \frac{u}{v}$.

Sol: G.T, $x = uv$

$$y = \frac{u}{v}.$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = uv.$

$$\frac{\partial x}{\partial u} = v. \quad , \quad \frac{\partial x}{\partial v} = u.$$

$$y = \frac{u}{v}.$$

$$\frac{\partial y}{\partial u} = \frac{1}{v} \quad , \quad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$$

$$J = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v}.$$

$$J = \frac{-2u}{v}.$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$x+y = uv + \frac{u}{v}$$

$$\frac{x}{y} = \frac{uv^2}{u} = v^2$$

$$v = \sqrt{\frac{x}{y}}$$

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$$x+y = u\left[v+\frac{1}{v}\right]$$

$$u = \frac{x+y}{v+\frac{1}{v}} = \frac{x+y}{\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}} = \frac{x+y}{\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}}$$

$$\frac{\partial u}{\partial x} = \left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right) - (x+y)\left[\frac{1}{2\sqrt{xy}}\right.$$

$$J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J' = \begin{vmatrix} \frac{1}{2\sqrt{xy}}\cdot y & \frac{1}{2\sqrt{\frac{x}{y}}} \times \frac{1}{y} \\ \frac{1}{2\sqrt{xy}}\cdot x & \frac{1}{2\sqrt{\frac{x}{y}}} \times x\left(\frac{-1}{y^2}\right) \end{vmatrix}$$

$$= \left(\frac{-1}{4y} - \frac{1}{4y}\right)$$

$$= \frac{-2}{4y} = \frac{-1}{2y}$$

$$JJ' = \frac{-2u}{v} \times \frac{-v}{2u}$$

$$= 1$$

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Type - 2

Q) S.T the functions $u = xe^y \sin z$, $v = xe^y \cos z$, $w = x^2 e^{2y}$ are functionally dependent and hence find the relation b/w them.

Sol: G.T, $u = xe^y \sin z$

$v = xe^y \cos z$

$w = x^2 e^{2y}$.

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = e^y \sin z$, $\frac{\partial u}{\partial y} = xe^y \sin z$, $\frac{\partial u}{\partial y} = xe^y \cos z$

$\frac{\partial v}{\partial x} = e^y \cos z$, $\frac{\partial u}{\partial y} = xe^y \cos z$, $\frac{\partial v}{\partial z} = -xe^y \sin z$

$\frac{\partial w}{\partial x} = 2xe^{2y}$, $\frac{\partial w}{\partial y} = 2x^2 e^{2y}$, $\frac{\partial w}{\partial z} = 0$.

$$J = \begin{vmatrix} e^y \sin z & xe^y \sin z & xe^y \cos z \\ e^y \cos z & xe^y \cos z & -xe^y \sin z \\ 2xe^{2y} & 2x^2 e^{2y} & 0 \end{vmatrix}$$

$$= \left\{ e^y e^y (2x) e^{2y} . x . x \right. \begin{vmatrix} \sin z & \sin z & \cos z \\ \cos z & \cos z & -\sin z \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 2e^{4y} x^3 \left[(-\sin^2 z - \cos^2 z) - (\sin^2 z - \cos^2 z) \right]$$

$$= 0.$$

Here the Jacobian, $J = 0$

$\therefore$ u, v & w are functionally dependent.

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We can form a relation b/w u, v & w.

(A)
$$u^2 = x^2 e^{2y} \sin^2 z$$
$$v^2 = x^2 e^{2y} \cos^2 z$$
$$w = x^2 e^{2y}$$
$$u^2+v^2 = x^2 e^{2y}(\sin^2 z + \cos^2 z)$$
$$u^2+v^2 = w.$$

which is the functional relation b/w u, v & w.

Q) S.T, $u = \sin^{-1}(x) + \sin^{-1}(y)$., $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ are functionally dependent, Hence find the relation b/w u & v.

Sol: G.T, $u = \sin^{-1}x + \sin^{-1}y$.
$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2}}$$

$$\frac{\partial v}{\partial x} = \sqrt{1-y^2} + \frac{y(-2x)}{2\sqrt{1-x^2}}, \quad \frac{\partial v}{\partial y} = \frac{x(-2y)}{2\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \frac{-xy}{\sqrt{1-x^2}} + \sqrt{1-y^2} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

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$$J = \left(\frac{-xy}{\sqrt{1-y^2}\sqrt{1-x^2}} + 1\right) - \left(1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right) = 0.$$

u, v are functionally dependent.

we can form relation b/w u & v.

$$\sin u = \sin(\sin^{-1}x + \sin^{-1}y)$$

$$= \left[\sin(\sin^{-1}x)\cos(\sin^{-1}x) + \cos(\sin^{-1}x)\sin(\sin^{-1}y)\right]$$

$$\sin u = x\cos(\sin^{-1}y) + y\cos(\sin^{-1}x)$$

$$\sin u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\sin u = v$$

$$u = \sin^{-1}v$$ //

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Q) Check whether the functions $u = \frac{x+y}{1-xy}$, $V = \tan^{-1}(x) + \tan^{-1}(y)$ are functionally dependent if so find the relation b/w them.

Sol: G.T, $u = \frac{x+y}{1-xy}$, $V = \tan^{-1}x + \tan^{-1}y$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)1 - (x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)1 - (x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$J = 0$.

Here the Jacobian $J = 0$

$\therefore$ u & v are functionally dependent.

$V = \tan^{-1}x + \tan^{-1}y$

$V = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$V = \tan^{-1}(u)$

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Q) P.T, $u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$ are functionally dependent. Hence find the relation b/w them.

Sol: G.T, $u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{(x-y)(1)-(x+y)(1)}{(x-y)^2} = \frac{x-y-x-y}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x-y)(1)-(x+y)(-1)}{(x-y)^2} = \frac{x-y+x+y}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x-y)^2 y-(xy)(2(x-y))}{(x-y)^4} = \frac{yx^2+y^3-2y^2x-2x^2y+2xy^2}{(x-y)^4}$$

$$= \frac{y^3+yx^2-2x^2y}{(x-y)^4}$$

$$\frac{\partial v}{\partial y} = \frac{(x-y)^2(x)-(xy)(2(x-y)(-1))}{(x-y)^4} = \frac{x^3+y^3x-2xy^2}{(x-y)^4}$$

$$J = \begin{vmatrix} \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \\ \frac{y^3+yx^2-2x^2y}{(x-y)^4} & \frac{x^3+y^3x-2xy^2}{(x-y)^4} \end{vmatrix}$$

$$= \left(\frac{-2yx^3-2y^3x+4xy^3}{(x-y)^6}\right) - \left(\frac{2xy^3+2x^3y-4x^3y}{(x-y)^6}\right)$$

$$= \frac{-2yx^3+2y^3x-2xy^3+2x^3y}{(x-y)^6} = 0$$

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Q). P.T, $u = xy+yz+zx$, $v = x^2+y^2+z^2$, $w = x+y+z$. are functionally dependent. Hence find the relation b/w them.

Sol: G.T, $u = xy+yz+zx$.

$v = x^2+y^2+z^2$

$w = x+y+z$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

$\frac{\partial w}{\partial x} = 1$, $\frac{\partial w}{\partial y} = 1$, $\frac{\partial w}{\partial z} = 1$

$\frac{\partial v}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = 2y$, $\frac{\partial v}{\partial z} = 2z$.

$u = xy+yz+zx$.

$\frac{\partial u}{\partial x} = y+z$, $\frac{\partial u}{\partial y} = x+z$, $\frac{\partial u}{\partial z} = x+y$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$C_2 \to C_2 - C_1$ $C_3 \to C_3 - C_1$

$$= \begin{vmatrix} y+z & x-y & x-z \\ 2x & 2(y-x) & 2(z-x) \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 2(x-y)(x-z) \begin{vmatrix} y+z & 1 & 1 \\ x & -1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$= 0$.

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Here the Jacobian $J=0$.

$\therefore$ u, v & w are functionally dependent.

$$w^2 = (x+y+z)^2$$

$$w^2 = x^2+y^2+z^2+2(xy+yz+zx)$$

$$w^2 = v+2u.$$

Q) If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = x^3+y^3+z^3-3xy$ then P.T. Jacobian $J=0$. Hence find the relation b/w u, v, w.

Sol: G.T, $u = x+y+z.$

$$v = x^2+y^2+z^2$$

$$w = x^3+y^3+z^3-3xyz.$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial w} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2y, \quad \frac{\partial v}{\partial z} = 2z$$

$$\frac{\partial w}{\partial x} = 3x^2-3yz, \quad \frac{\partial w}{\partial y} = 3y^2-3xz, \quad \frac{\partial w}{\partial z} = 3z^2-3xy$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2-yz & 3y^2-2x & 3z^2-yx \end{vmatrix}$$

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$$= 2x^3\left[6y(z^2-xy)-6z(y^2-zx)\right]-1\left[6x(z^2-yx)-6z(x^2-yz)\right]$$

$$+1\left[6x(y^2-xz)-6y(x^2-yz)\right]$$

$$= 6\left[z^2y-xy^2-y^2z+xz^2-xz^2+x^2y+z^2z-z^2y+y^2x-x^2z+y^2z...\right]$$

$$=0$$

Type - 3

Q) If $x=u+uv$, $y=v+uv$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol: G.T, $x=u+uv$, $y=v+uv$.

Here x & y functions of u & v.

we have to find $\frac{\partial(u,v)}{\partial(x,y)}$

It is difficult to express u & v in terms of x & y. So we find $\frac{\partial(x,y)}{\partial(u,v)}$

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$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\frac{\partial x}{\partial u} = 1+v. \quad , \quad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = v \quad , \quad \frac{\partial y}{\partial v} = 1+u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+u)(1+v) - uv = 1+u+v.$$

W.K.T, $JJ' = 1$. i.e, $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \frac{1}{1+u+v.}$$

Q) If $x = uv$, $y = \frac{u+v}{u-v}$. Find $\frac{\partial(u,v)}{\partial(x,y)}$.

Sol: G.T, $x = uv$

$y = v+uv.$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = \frac{(u-v)1 - (u+v)(1)}{(u-v)^2} = \frac{-2v}{(u-v)^2}$$

$$\frac{\partial y}{\partial v} = \frac{(u-v)(1) - (u+v)(-1)}{(u-v)^2} = \frac{u-v+u+v}{(u-v)^2} = \frac{2u}{(u-v)^2}$$

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$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix}$$

$$= \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2}$$

$$= \frac{4uv}{(u-v)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}.$$

Type-4:

Q) If $x+y+z=u$, $y+z=uv$, $z=uvw$. Then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Sol: G.T, $x+y+z=u$, $y+z=uv$.

$uvw = z$.

we have to find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

$z = uvw$, $y = uv - z$.

$y = uv - uvw$

$x = u - y - z$

$x = u - (uv - uvw) - uvw$

$x = u - uv$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

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$$\frac{\partial x}{\partial u} = 1-v \qquad \frac{\partial y}{\partial u} = v-vw \qquad \frac{\partial z}{\partial u} = vw$$

$$\frac{\partial x}{\partial v} = -u \qquad \frac{\partial y}{\partial v} = u-uw. \qquad \frac{\partial z}{\partial v} = uw$$

$$\frac{\partial x}{\partial w} = 0 \qquad \frac{\partial z}{\partial w} = -uv \qquad \frac{\partial z}{\partial w} = uv.$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= u.uv \begin{vmatrix} 1-v & -1 & 0 \\ v-vw & 1-w & -1 \\ vw & w & 1 \end{vmatrix} \quad R_2 \to R_2+R_3$$

$$= u^2v \begin{vmatrix} 1-v & -1 & 0 \\ v & 1 & 0 \\ vw & w & 1 \end{vmatrix} = u^2v$$

Q) If $u = x+2y^2-z^3$, $v = 2x^2yz$, $w = 2z^2-xy$. (Ans: 10)

Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at the point $(1,-1,0)$

Sol: G.T., $u = x+2y^2-z^3$

$v = 2x^2yz$

$w = 2z^2-xy.$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\frac{\partial y}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 4y, \quad \frac{\partial u}{\partial z} = -3z^2$$

$$\frac{\partial v}{\partial x} = 4xyz, \quad \frac{\partial v}{\partial y} = 2x^2z, \quad \frac{\partial v}{\partial z} = 2x^2y.$$

$$\frac{\partial z}{\partial x} = -y, \quad \frac{\partial z}{\partial y} = -x, \quad \frac{\partial z}{\partial z} = 4z.$$

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$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 4y & -3z^2 \\ 4xyz & 2x^2z & 2x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -4 & 0 \\ 0 & 0 & -2 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1(-2) + 4(2)$$

$$= 8 - 2 = 6.$$

Q) If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Sol: G.T, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$ $\frac{\partial z}{\partial r} = 0$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$. $\frac{\partial y}{\partial \theta} = r\cos\theta$ $\frac{\partial z}{\partial \theta} = 0$

$\frac{\partial x}{\partial z} = 0$. $\frac{\partial y}{\partial z} = 0$. $\frac{\partial z}{\partial z} = 1$

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$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Q) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Sol: G.T, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$, $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$, $\frac{\partial x}{\partial \phi} = r\sin\theta(-\sin\phi)$

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$, $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$, $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$

$\frac{\partial z}{\partial r} = \cos\theta$, $\frac{\partial z}{\partial \theta} = r(-\sin\theta)$, $\frac{\partial z}{\partial \phi} = 0$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi(-r\sin\theta\cos\theta\cos\phi)$$

$$-r\sin\theta\cos\phi(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi).$$

Q) If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$.

Find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Sol: $x = r\cos\theta$, $y = r\sin\theta$

G.T, $u = x^2 - y^2$, $v = 2xy$

$u = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta.$

$v = 2xy = 2r\cos\theta \cdot r\sin\theta = r^2\sin 2\theta.$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix}$$

$\frac{\partial u}{\partial r} = 2r\cos 2\theta$, $\frac{\partial u}{\partial \theta} = 2r^2(-\sin 2\theta)$

$\frac{\partial v}{\partial r} = 2r\sin 2\theta$, $\frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta.$

$$= \begin{vmatrix} 2r\cos 2\theta & 2r^2(-\sin 2\theta) \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}$$

$= 4r^3\cos^2 2\theta + 4r^3\sin^2 2\theta$

$= 4r^3.$

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The chain rule of partial differentiation.

If $z = f(x,y)$, where $x = \phi(t)$, $y = \psi(t)$.

then z is called a Composite function of a variable 't'.

If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$ then z is called a Composite function of two variables u & v.

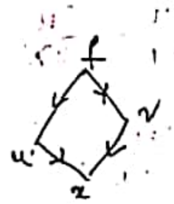
i) If $f(u)$ is a differentiable function of a variable u and $u = u(x)$ is also a differentiable function.

Then we have the chain rule :

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}$$

ii) Let $f(u,v)$ be a differentiable function of two independent variables u & v. Let u, v be differentiable functions of the independent variable x.

i.e. $u = u(x)$, $v = v(x)$.



Then we have the chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x}.$$

iii) If $f(u,v)$ is a differentiable function of u & v, u & v are also differentiable functions of two independent variables x & y, then the partial derivatives of f w.r.t x & y are given by the Chain rule.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

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Total differential coefficient:

Let $z = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$; substituting x & y in z, z becomes a function of single variable, then the derivative of z w.r.t 't' i.e $\frac{dz}{dt}$ is called total differential coefficient (or) total derivative of z.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

It can be written as $\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$.

Q) If $u = f(x-y, y-z, z-x)$. P.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Sol: G.T, $u = f(x-y, y-z, z-x)$.

Let $r = x-y$

$s = y-z$

$t = z-x$.

Then 'u' becomes $u = f(r,s,t)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$\frac{\partial r}{\partial x} = 1$, $\cancel{\frac{\partial u}{\partial x}} =$

$\frac{\partial s}{\partial x} = 0$

$\frac{\partial t}{\partial x} = -1$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y}.$$

$\frac{\partial r}{\partial y} = -1$, $\frac{\partial s}{\partial y} = 1$, $\frac{\partial t}{\partial y} = 0$.

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$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} = \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} \quad \text{—②}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}.$$

$$\frac{\partial r}{\partial z} = 0, \quad \frac{\partial s}{\partial z} = -1, \quad \frac{\partial t}{\partial z} = 1.$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s}. \quad \text{—③}$$

①+②+③

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s} = 0.$$

Q) If $z = f(x, y)$ where $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$..

S.T $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$..

Sol: G.T, $z = f(x, y)$

Let ~~$u = x$~~ $\quad x = e^{u} + e^{-v}$

~~$v = y$~~ $\quad y = e^{-u} - e^{v}$

~~Then 'z' becomes $z = f(u, v)$~~

~~$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} +$~~

$z = f(x, y)$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \quad \text{— ①.}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y} \quad \text{—②}$$

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from ① - ②.

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=e^{u}\frac{\partial z}{\partial x}-e^{-u}\frac{\partial z}{\partial y}+e^{+v}\frac{\partial z}{\partial x}-e^{+v}\frac{\partial z}{\partial y}$$

$$=\frac{\partial z}{\partial x}\left(e^{u}+e^{-v}\right)-\frac{\partial z}{\partial y}\left(e^{-u}-e^{v}\right)$$

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$

Q) If $u=f(r)$ and $x=r\cos\theta$, $y=r\sin\theta$, P.T

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

Sol: G.T, $u=f(r)$, $x=r\cos\theta$, $y=r\sin\theta$.

$$x^2+y^2=r^2 \Rightarrow r=\sqrt{x^2+y^2}.$$

$$u=f'(r)$$

diff w.r.t 'x', we get

$$\frac{\partial u}{\partial x}=f'(r)\cdot\frac{\partial r}{\partial x}$$

diff w.r.t 'x', we get

$$\frac{\partial^2 u}{\partial x^2}=f''(r)\left(\frac{\partial r}{\partial x}\right)^2+f'(r)\frac{\partial^2 r}{\partial x^2}$$

Illy $\frac{\partial^2 u}{\partial y^2}=f''(r)\left(\frac{\partial r}{\partial y}\right)^2+f'(r)\frac{\partial^2 r}{\partial y^2}$

$$\cancel{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)\left(\frac{\partial r}{\partial z}\right)^2+f'(r)\frac{\partial^2 r}{\partial z^2}}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]+f'(r)\left[\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}\right]$$

$$r^2=x^2+y^2$$

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$$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - \frac{x^2}{r}}{r^2} \Rightarrow \frac{r^2 - x^2}{r^3}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}$$

$$\text{IIly } \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(x)\left[\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right]$$

$$= f''(x) + \frac{1}{r} f'(r).$$

Q) $x = r\cos\theta$, $y = r\sin\theta$. S.T $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ & $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$

Sol: G.T., $\cancel{\frac{1}{r}\frac{\partial x}{\partial \theta}} = \cancel{r\frac{\partial \theta}{\partial x}}$

$x = r\cos\theta$, $y = r\sin\theta$

$$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$$

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$x = r\cos\theta$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -r\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$$

$$\frac{1}{r}\left(\frac{-ry}{\sqrt{x^2+y^2}}\right) = \frac{-ry}{x^2+y^2} = \frac{-\sqrt{x^2+y^2}\,y}{x^2+y^2}$$

$$\frac{-y}{\sqrt{x^2+y^2}} = \frac{-y}{\sqrt{x^2+y^2}}$$

$$\therefore \quad \frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$$

Q) If $u = x\log xy$ where $x^3+y^3+3xy=1$. Find $\frac{\partial u}{\partial x}$.

Sol: G.T, $u = x\log xy$

$x^3+y^3+3xy=1$ (we treat y is a function of single variable x).

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\,1 + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x}.$$

$x^3+y^3+3xy=1$

diff. w.r.t 'x', we get.

$3x^2+3y^2\cdot y' + 3y + 3xy' = 0.$

$x^2+y'(y^2+x)+y=0$

$$y' = \frac{-(y+x^2)}{y^2+x}.$$

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$u = x \log xy$

$$\frac{\partial u}{\partial x} = x \cdot \frac{1}{xy}(y) + \log xy \cdot 1$$

$$\frac{\partial u}{\partial x} = 1 + \log xy$$

$$\frac{\partial u}{\partial y} = x\left(\frac{1}{xy}\right)x = \frac{x}{y}$$

$$\frac{du}{dx} = (1+\log xy) + \frac{x}{y}\left(-\frac{(x^2+y)}{y^2+x}\right)$$

$$\frac{du}{dx} = 1+\log xy - \frac{x(y+x^2)}{y(y^2+x)}$$

Q) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, P.T $x\frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = 0.$

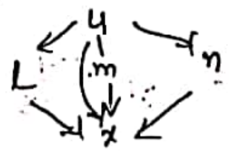
Sol: Let $\frac{x}{y} = l$, $\frac{y}{z} = m$, $\frac{z}{x} = n$.

$$\frac{\partial l}{\partial x} = \frac{1}{y}, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = \frac{-z}{x^2}$$

$$\frac{\partial l}{\partial y} = \frac{-x}{y^2}, \quad \frac{\partial m}{\partial y} = \frac{1}{z}, \quad \frac{\partial n}{\partial y} = 0$$

$$\frac{\partial l}{\partial z} = 0, \quad \frac{\partial m}{\partial z} = \frac{-y}{z^2}, \quad \frac{\partial n}{\partial z} = \frac{1}{x}.$$

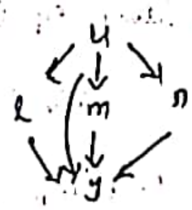
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial x}$$



$$= \frac{\partial u}{\partial l} \cdot \frac{1}{y} + \frac{\partial u}{\partial m} \cdot 0 + \frac{\partial u}{\partial n}\left(\frac{-z}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y} \cdot \frac{\partial u}{\partial l} - \frac{z}{x} \cdot \frac{\partial u}{\partial n} \quad \text{—(1).}$$

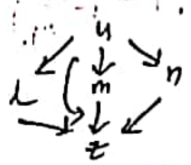
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \frac{\partial m}{\partial y} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial y}$$



$$= \frac{\partial u}{\partial l}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial m} \cdot \frac{1}{z} + \frac{\partial u}{\partial n} \cdot 0.$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{y} \cdot \frac{\partial u}{\partial l} + \frac{y}{z} \cdot \frac{\partial u}{\partial m} \quad \text{—(2).}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \frac{\partial m}{\partial z} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial z}$$



$$= \frac{\partial u}{\partial l} \cdot 0 + \frac{\partial u}{\partial m}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial n} \cdot \frac{1}{x}$$

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$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\cdot\frac{\partial u}{\partial m} + \frac{z}{x}\cdot\frac{\partial u}{\partial n} \quad —③.$$

Adding ①, ② & ③.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Q) If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$ S.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol: $l = e^{y-z}$ $\quad m = e^{z-x}$ $\quad n = e^{x-y}$

$\frac{\partial l}{\partial x} = 0$ $\quad \frac{\partial m}{\partial x} = -e^{z-x} = -m$ $\quad \frac{\partial n}{\partial x} = e^{x-y} = n$

$\frac{\partial l}{\partial y} = e^{y-z} = l$ $\quad \frac{\partial m}{\partial y} = 0$ $\quad \frac{\partial n}{\partial y} = -e^{x-y} = -n$

$\frac{\partial l}{\partial z} = -e^{y-z} = -l$ $\quad \frac{\partial m}{\partial z} = e^{z-x} = m$ $\quad \frac{\partial n}{\partial z} = 0.$

$$u = f(e^{y-z}, e^{z-x}, e^{x-y}) = f(l, m, n)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}(-m) + \frac{\partial u}{\partial n}\cdot n = -m\frac{\partial u}{\partial m} + n\frac{\partial u}{\partial n} \quad —①$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}\cdot l + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}(-n) = l\frac{\partial u}{\partial l} - n\frac{\partial u}{\partial n} \quad —②$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= \frac{\partial x}{\partial l}(-l) + \frac{\partial u}{\partial m}\cdot m + \frac{\partial u}{\partial n}\cdot 0 = -l\frac{\partial u}{\partial l} + m\frac{\partial u}{\partial m} \quad —③$$

Adding eq ①, ② & ③.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Q) If $x = e^u \operatorname{cosec} v$, $y = e^u \cot v$ then S.T,

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$

Sol: $z = f(x,y)$, $x = e^u \operatorname{cosec} v$, $y = e^u \cot v$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u \operatorname{cosec} v + \frac{\partial z}{\partial y} e^u \cot v.$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

$$= \frac{\partial z}{\partial x}\left(-e^u \operatorname{cosec} v\cdot\cot v\right) + \frac{\partial z}{\partial y}\left(-e^u \operatorname{cosec}^2 v\right)$$

$$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right] = e^{-2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\operatorname{cosec}^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cot^2 v\right.$$

$$+ 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot e^{2u}\operatorname{cosec} v\cot v$$

$$+ (-\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2\left(e^{2u}\operatorname{cosec}^2 v\cot^2 v\right) + (-\sin^2 v)\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\operatorname{cosec}^4 x$$

$$\left. + (-\sin^2 v)\, 2\,\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\, e^{2u}\operatorname{cosec}^3 v\, \cot v\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2(\operatorname{cosec}^2 v - \cot^2 v) + \left(\frac{\partial z}{\partial y}\right)^2(\cot^2 v - \operatorname{cosec}^2 v)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2.$$

9)

Sol: $l = \dfrac{y-x}{xy} = \dfrac{1}{x} - \dfrac{1}{y}$, $m = \dfrac{z-x}{xz} = \dfrac{1}{x} - \dfrac{1}{z}.$

$\dfrac{\partial l}{\partial x} = \dfrac{1}{x^2}$ $\qquad$ $\dfrac{\partial m}{\partial x} = \dfrac{-1}{x^2}$

$\dfrac{\partial l}{\partial y} = \dfrac{-1}{y^2}$ $\qquad$ $\dfrac{\partial m}{\partial y} = 0.$

$\dfrac{\partial l}{\partial z} = 0$ $\qquad$ $\dfrac{\partial m}{\partial z} = \dfrac{1}{z^2}$

$$u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right) = f(l,m)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} = \frac{\partial u}{\partial l}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial m}\left(\frac{-1}{x^2}\right)$$

$$x^2\frac{\partial u}{\partial x} = -\left(\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\right) \quad \text{--- ①}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}$$

$$= \frac{\partial u}{\partial l}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial m}\cdot 0.$$

$$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \quad \text{--- ②}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}\left(\frac{1}{z^2}\right)$$

$$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial m} \quad \text{--- ③}$$

Adding ①, ② & ③

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\left(\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\right) + \frac{\partial u}{\partial l} + \frac{\partial u}{\partial m} = 0$$

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Maxima and Minima function of two Variables:

Let $f(x,y)$ be a function of two Variables be x & y. At $x=a$, $y=b$, $f(x,y)$ is said to have maximum or minimum Value if $f(a,b) > f(a+h, b+k)$ or $f(a,b) < f(a+h, b+k)$ respectively where h & k are small values.

Extreme Value:

$f(a,b)$ is said to be an extreme Value of f if it is a maximum or minimum Value.

i) Necessary Conditions: ~~for $f(x,y)$~~

The necessary Conditions for $f(x,y)$ to have max or min Value at a,b are

$$f_x(a,b)=0, \quad f_y(a,b)=0$$

ii) Sufficient Conditions:

Suppose that $f_x(a,b)=0$, $f_y(a,b)=0$ and

Let
$$\frac{\partial^2 f}{\partial x^2}(a,b) = r$$
$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = s.$$
$$\frac{\partial^2 f}{\partial y^2}(a,b) = t.$$

then i) $f(a,b)$ is a max Value if $rt-s^2>0$ and $r<0$

ii) $f(a,b)$ is a min value if $rt-s^2>0$ and $r>0$.

iii) $f(a,b)$ is not an extreme Value if $rt-s^2<0$

iv) If $rt-s^2=0$ then $f(x,y)$ fails to have maximum or minimum value and it needs

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further investigation.

Stationary value :-

$f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b)=0$, $f_y(a,b)=0$. Thus every extreme value is a stationary value. But the converse may not be true.

Saddle point :-

A point (a,b) is said to be saddle point of $f(x,y)$ if $rt-s^2<0$ or if $f(x,y)$ is not an extreme value.

Working procedure :-

Let $f(x,y)$ be a function of two variables x & y

Step-1 : differentiate $f(x,y)$ w.r.t x & y partially, we get

f_x (or) $\frac{\partial f}{\partial x}$; f_y (or) $\frac{\partial f}{\partial y}$

Step-2 : Equate $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ to zero, we get

$\frac{\partial f}{\partial x}=0$ — ① , $\frac{\partial f}{\partial y}=0$ — ②.

Step-3 : Solving eq ① & ②, we get the stationary points $(a_1,b_1)\cdot(a_2,b_2)$ ----

Step-4 : Find r,s,t

$$r=\frac{\partial^2 f}{\partial x^2};\quad s=\frac{\partial^2 f}{\partial x\,\partial y},\quad t=\frac{\partial^2 f}{\partial y^2}.$$

Step-5 : Case-i.

At the point (a_1,b_1)

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Find the Values of r, s & t at the point a_1, b_1

i) If $rt - s^2 > 0$ & $r < 0$ then f is maximum at (a_1, b_1) and the maximum Value is $f(a_1, b_1)$.

ii) If $rt - s^2 > 0$ & $r > 0$ then f is minimum at (a_1, b_1) and the minimum Value is $f(a_1, b_1)$.

iii) If $rt - s^2 < 0$ then f is neither maximum nor minimum at (a_1, b_1)

iv) If $rt - s^2 = 0$. no conclusion can drawn.

Case - ii :

At the point (a_2, b_2)

we proceed like case ①.

Q) Find an extreme Values of the function

$f = x^3 + y^3 - 63x - 63y + 12xy$.

Sol: G.T, $f = x^3 + y^3 - 63x - 63y + 12xy$ —— ①

Step-① diff ① w.r.t x & y partially, we get

$$f_x = \frac{\partial f}{\partial x} = 3x^2 - 63 + 12y$$

$$f_y = \frac{\partial f}{\partial y} = 3y^2 - 63 + 12x$$

Step-② Equate $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ to zero, we get.

$f_x = 0$ i.e $3x^2 - 63 + 12y = 0$

$x^2 - 21 + 4y = 0$ —— ②

$f_y = 0$ i.e $3y^2 - 63 + 12x = 0$

$y^2 - 21 + 4x = 0$ —— ③

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Step - ② Solving eq ② & ③.

②−③ gives, $x^2-y^2+4y-4x=0$.

$$(x-y)(x+y)+4(y-x)=0.$$

$$(x-y)(x+y-4)=0.$$

$$x-y=0 \quad \text{—④} \qquad x+y-4=0 \quad \text{—⑤}.$$

Step. Solving ⑤ & ④, we get

$$y^2+4y-21=0.$$

$$y=-7,3$$

when $y=-7$, $x=-7$ [$\because$ from ④]

when $y=3$, $x=3$

From ② & ⑤, we get [$\because y=4-x$]

$$x^2+4(4-x)-21=0$$

$$x^2-4x-5=0$$

$$x=5,-1.$$

when $x=5$, $y=-1$ [$\because$ from ⑤]

when $x=-1$, $y=5$

$\therefore$ The stationary points are

$P_1(-7,-7)$ $P_2(3,3)$ $P_3(5,-1)$ $P_4(-1,5)$.

$$r=\frac{\partial^2 f}{\partial x^2}=6x.$$

$$s=\frac{\partial^2 f}{\partial x\,\partial y}=12.$$

$$t=\frac{\partial^2 f}{\partial y^2}=6y.$$

Case ①: At the point $P_1(-7,-7)$

$$r=6x=-42<0.$$

$$s=12$$

$$t=6y=-42.$$

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$rt-s^2 = (-42)(-42) - 12^2 = 1764 - 144 = 1620$

Here, $r<0$ and $rt-s^2>0$

$\therefore$ f is maximum at the point $(-7,-7)$.

$f_{max} = (-7)^3+(-7)^3-63(-7)-63(-7)+12(-7)(-7) = 784$

<u>Case-②</u>: At the point $P_2(3,3)$

$r = 6x = 18>0$

$s = 12$

$t = 6y = 18$

$rt-s^2 = 18^2-12^2 = 180>0.$

Here $r>0$ & $rt-s^2>0.$

$\therefore$ f is minimum at the point $(3,3)$.

$f_{min} = (3)^3+(3)^3-63(3)-63(3)+12(3)(3) = -216.$

<u>Case-③</u>: At the point $P_3(5,-1)$.

$r = 6x = 30.$

$s = 12$

$t = 6y = -6$

$rt-s^2 = -180-144 = -324<0.$

Here $rt-s^2<0.$ ~~& $r>0$.~~

$\therefore$ f is neither max nor min at the point $(5,-1)$.

<u>Case-④</u>: At the point $P_4(-1,5)$

$r = 6x = -6.$

$s = 12$

$t = 6y = 30.$

$rt-s^2 = -180-144 = -324<0$

Here $rt-s^2<0$

$\therefore$ f is neither max nor min at the point $(-1,5)$.

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$\therefore$ Maximum Value of f at the point $(-7,-7)$ is 784

$\therefore$ Minimum Value of f at the point $(3,3)$ is -216

8) Find the max & min values of the function $f = x^3+3xy^2-3x^2-3y^2+4$.

Sol: G.T, $f = x^3+3xy^2-3x^2-3y^2+4$. — ①

Step-①: diff ① w.r.t x & y partially.

$$f_x = \frac{\partial f}{\partial x} = 3x^2+3y^2-6x$$

$$f_y = \frac{\partial f}{\partial y} = 6xy-6y$$

Step-②: Equate $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ to zero

$f_x=0$. i.e $3x^2+3y^2-6x=0$.

$x^2+y^2-2x=0$ — ②.

$f_y=0$ i.e $6xy-6y=0$

$xy-y=0$ — ③.

Step-③: Solving eq ② & ③, we get

$x=0,1,2$

$y=0,\pm 1$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x-6$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x-6.$$

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Case-①

At $P_1(0,0)$

$r = -6$

$s = 0$

$t = -6$

$rt - s^2 = 36 > 0$

Here $r < 0$, $rt - s^2 > 0$

$\therefore$ f is max at $P_1(0,0)$

$f_{max} = 4$

Case-② At $P_2(2,0)$

$r = 6$

$s = 0$

$t = 6$

$rt - s^2 = 36 > 0$

Here $r > 0$, $rt - s^2 > 0$

$\therefore$ f is min at $P_2(2,0)$

$f_{min} = 8 - 12 + 4 = 0$

Case-③ At $(1, \pm 1)$

$r = 0$

$s = \pm 6$

$t = 0.$

$rt - s^2 = -36 < 0$

$\therefore$ f is not an extreme value.

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Q) Find the extremum values of the function $\sin x, \sin y, \sin(x+y)$ where $0 < x < \pi$, $0 < y < \pi$,

Sol: G.T, $f = \sin x \sin y \sin(x+y)$. —— ①.

Step – 1 : diff ① w.r.t x & y partially.

Here $f_x = \frac{\partial f}{\partial x} = \sin y\left[\cos x \sin(x+y) + \sin x (\cos(x+y)\right]$

$\frac{\partial f}{\partial x} = \sin y \sin(2x+y)$

$f_y = \frac{\partial f}{\partial y} = \sin x\left[\cos y \sin(x+y) + \sin y \cos(x+y)\right]$

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$$\frac{\partial f}{\partial y} = \sin x \sin(x+2y)$$

Step-②: Equate $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ to zero, we get

$f_x = 0$ i.e $\sin y \sin(2x+y) = 0$.

$\sin y \neq 0$ for $0 < y < \pi$.

$\sin(2x+y) = 0$

$2x + y = 0$ —②

$\because 2x+y \neq 0$ for $0<x<\pi$, $0<y<\pi$

$f_y = 0$ i.e $\sin x \sin(x+2y) = 0$

$\sin x \neq 0$ for $0 < y < \pi$

$\sin(x+2y) = 0$.

$x + 2y = \pi$ —③

Step-③: Solving ② & ③, we get.

$x = \frac{\pi}{3}$, $y = \frac{\pi}{3}$

$\therefore$ The stationary point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Step-④:

$$r = \frac{\partial^2 f}{\partial x^2} = 2\sin y(\cos(2x+y))$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \cos x \sin(x+2y) + \sin \cos(x+2y)$$

$$= \sin(2x+2y).$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2\sin x \cos(x+2y).$$

Step-⑤:

Case-i) At the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$r = 2\sin\frac{\pi}{3} * \cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right) = \cancel{\frac{\sqrt{3}}{2}}\ -\sqrt{3}$

$s = \sin\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$t = 2\sin\frac{\pi}{3}\cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) = -\sqrt{3}$

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$rt-s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$

Here $r<0$ & $rt-s^2>0$.

$\therefore$ f is maximum at the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$\therefore f_{max} = \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = \frac{3}{4}\left(\frac{1}{2}\right) = \frac{3}{8}$.

Q) Find the extremum values of the function
$f = \sin x + \sin y + \sin(x+y)$.

Sol: Given that, $f(x,y) = \sin x + \sin y + \sin(x+y)$ — ①

~~Ist~~ Step-①: diff ① w.r.t. x & y partially.

$f_x = \frac{\partial f}{\partial x} = \cos x + [\cos y \sin x + \cos x \sin y]$

$= \cos x + [\cos x \cos y + \sin x \sin y]$

$= \cos x + \cos(x+y)$ — ②

$f_y = \frac{\partial f}{\partial y} = \cos y + \cos(x+y)$ — ③.

~~Conside~~ Step-②: Equate $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ to 0, we get $\cos x = \cos y$
$x = y$

$f_x = 0$ i.e $\cos x + \cos(x+y) = 0$

$\cos x + \cos 2x = 0$

$= 2\cos\frac{3x}{2}\cos\frac{x}{2} = 0$

$\Rightarrow \frac{3x}{2} = \pm\frac{\pi}{2}$ (or) $\frac{x}{2} = \pm\frac{\pi}{2}$

$\Rightarrow x = \pm\frac{\pi}{3}$ (or) $x = \pm\pi$

$\therefore x = \pm\frac{\pi}{3}$, $y = \pm\frac{\pi}{3}$ i.e $\left(\pm\frac{\pi}{3}, \pm\frac{\pi}{3}\right)$

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and $x = \pm\pi$, $y = \pm\pi$ i.e $(\pm\pi, \pm\pi)$.

Step-④: $r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$

$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

Step-⑤: At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$l = -\sqrt{3}$, $m = -\frac{\sqrt{3}}{2}$ and $n = -\sqrt{3}$

$\therefore$ $rt - s^2 = \frac{9}{4} > 0$, and $r < 0$.

f is maximum at $\left[\frac{\pi}{3}, \frac{\pi}{3}\right]$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $f = \frac{3\sqrt{3}}{2}$

$\therefore$ Maximum Value of $f = \frac{3\sqrt{3}}{2}$.

We can prove that $rt - s^2$ is positive and r is positive at $\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$

$\therefore$ f has a minimum at $\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$

Minimum value of $v = -\frac{3\sqrt{3}}{2}$.

At $(\pm\pi, \pm\pi)$, $rt - s^2 = 0$.

There is a need for investigation

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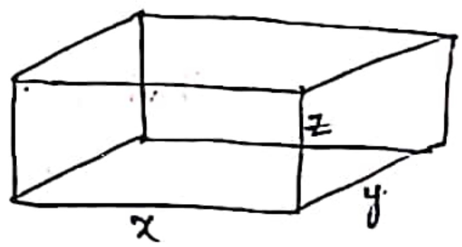
Q) A rectangular box opened at the top is to have volume of 32 cubic units. Find the dimensions of the box least material for its construction.

Sol: Let x, y & z ~~with~~ be the length, breadth & height of the rectangular box.

area of the bottom of rectangular box is xy



Area of the two sides (left & right) is $yz + yz = 2yz$

Area of the two sides (front & back) is $xz + xz = 2xz$.

Total surface area of the opened rectangular box is $S = xy + 2yz + 2zx$. — ①

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We have Volume $V = xyz$.

Given that, Volume $V = 32$

i.e, $xyz = 32$.

$$z = \frac{32}{xy} \quad \text{②}$$

From ① & ②.

$$S = xy + \frac{64}{y} + \frac{64}{x}$$

Let $f = xy + \frac{64}{x} + \frac{64}{y} \quad$ — ③.

Step-①: diff ③ w.r.t x & y partially, we get.

$$f_x = y - \frac{64}{x^2}$$

$$f_y = x - \frac{64}{y^2}$$

Step-②: Equate $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ to zero, we get.

$f_x = 0$ i.e, $y - \frac{64}{x^2} = 0$

$$y = \frac{64}{x^2} \quad \text{④}$$

$f_y = 0$ i.e, $x - \frac{64}{y^2} = 0$.

$$x = \frac{64}{y^2} \quad \text{⑤}$$

Step-③: Solving ④ & ⑤, we get.

$$y = \frac{64}{x^2}$$

$$y = \frac{64}{\left(\frac{64}{y^2}\right)^2} = \frac{64 \cdot y^4}{64 \cdot 64}$$

$$y^4 - 64y = 0.$$

$$y(y^3 - 64) = 0.$$

$$\cancel{y=0}, \; y = 4$$

We neglect $y=0$ because breadth cannot be zero.

when $y = 4$

$$x = \frac{64}{4^2} = 4 \quad \because \text{from } (5).$$

$\therefore$ The stationary point is $(4,4)$

Step-(4) :-

$$r = f_{xx} = \frac{128}{x^3}$$

$$s = f_{xy} = 1$$

$$t = f_{yy} = \frac{128}{y^3}.$$

Step-(5) :-

Case-i: At the point $(4,4)$

$$r = \frac{128}{x^3} = 2 > 0.$$

$$s = 1$$

$$t = \frac{128}{y^3} = 2$$

$$rt - s^2 = 4 - 1 = 3 > 0.$$

Here $r > 0$, $rt - s^2 > 0$

$\therefore$ f is minimum at the point $(4,4)$.

we have $z = \frac{32}{xy}$

$$z = \frac{32}{4(4)} = 2$$

$\therefore$ dimensions of the box: least material for its construction is $l = 4$, $b = 4$ & $h = 2$

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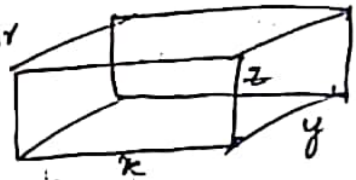
Q) A rectangular box opened at the top is to have volume of 120 cubic units. Find the dimensions of the box least material for its construction.

Sol: Let x, y, z be the l, b & h of rectangular box.

Area of the bottom of rectangular box is xy,



Area of the two sides (left & right) is $yz + yz = 2yz$

Area of the two sides (front & back) is $xz + xz = 2xz$.

Total surface area of the opened rectangular box is $S = xy + 2yz + 2zx$ — ①

we have volume $V = xyz$

G.T, $V = 120$

i.e $xyz = 120$

$$z = \frac{120}{xy} \quad — ②$$

From ① & ②

$$S = xy + \frac{240}{y} + \frac{240}{x}$$

$$\text{Let } f = xy + \frac{120}{x} + \frac{120}{y} \quad — ③$$

Step-①: diff ③ w.r.t x & y partially.

$$f_x = y - \frac{120}{x^2}$$

$$f_y = x - \frac{120}{y^2}$$

Step-②: equate $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ to zero.

$f_x = 0$ i.e $y - \frac{120}{x^2} = 0 \Rightarrow y = \frac{120}{x^2}$ — ④

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$f_y = 0$ i.e , $x - \frac{120}{y^2} = 0 \Rightarrow x = \frac{120}{y^2}$ — (5)

Step-(3): Solving (4) & (5) , we get

$$y = \frac{120}{x^2}$$

$$y = \frac{1\cancel{20}}{\left(\frac{120}{y^2}\right)^2} = \frac{120\, y^4}{120 \cdot 120}$$

$$y^4 - 120y = 0.$$

$$y(y^3 - 120) = 0$$

$$y = 0 \quad , \quad y = \sqrt[3]{120} = 2\sqrt{30}$$

we neglect $y=0$ because breadth cannot be zero.

when $y = 2\sqrt{30}$

$$x = \frac{12\cancel{0}}{4(3\cancel{0})} = 1$$

∴ The stationary point is $(1, 2\sqrt{30})$

Step-(4): $r = f_{xx} = 120\left(\frac{2}{x^3}\right) = \frac{240}{x^3}$

$$s = f_{xy} = 1$$

$$t = f_{yy} = \frac{240}{y^3}$$

Step-(5): Case-i: At point $(1, 2\sqrt{30})$

$$r = \frac{240}{x^3} = \frac{240}{1} > 0.$$

$$s = 1$$

$$t = \cancel{120}\ \frac{240}{y^3} = \frac{240}{240\sqrt{3}} = \frac{1}{\sqrt{3}} > 0$$

$$rt - s^2 = \frac{240}{\sqrt{3}} > 0$$

Here , $r > 0$, $rt - s^2 > 0$

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$\therefore f$ is minimum at point $(1, 2\sqrt{30})$

we have $z = \frac{120}{xy} = \frac{120}{2\sqrt{30}} = \frac{60}{\sqrt{30}}$.

$\therefore$ dimensions of the box least material for its construction, is $l = 1$, $b = 2\sqrt{30}$, $h = 2\sqrt{30}$.

Q) Find the point on the surface $z^2 = xy+1$ nearest to the origin.

Sol: Let $O(0,0,0)$ be the origin.

$P(x,y,z)$ be an arbitary point on surface $z^2 = xy+1$

$OP = \sqrt{x^2+y^2+z^2}$

$OP^2 = x^2+y^2+z^2$.

G.T, eq. of the surface is $z^2 = xy+1$. —①

$OP^2 = x^2+y^2+xy+1$ —② [$\because$ from ①].

Let $f = x^2+y^2+xy+1$ —③.

we have to minimize the function f satisfying the condition ①

Step-①: diff ③ w.r.t 'x & y' partially, we get

$f_x = 2x+y$

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$f_y = 2y + x$.

Step-② : Equate f_x & f_y to zero.

$f_x = 0$ i.e $2x + y = 0$ — ④

$f_y = 0$ i.e $2y + x = 0$. — ⑤.

Step-③ : Solving ④ & ⑤, we get

$x = 0$, $y = 0$.

∴ The stationary point is (0,0).

Step-④ : $r = f_{xx} = 2$

$s = f_{xy} = 1$

$t = f_{yy} = 2$

Step-⑤ : Case - i :

At the point (0,0)

$r = 2 > 0$

$s = 1$

$t = 2$

$rt - s^2 = 4 - 1 = 3 > 0$.

Here ~~r is~~ $r > 0$ & $rt - s^2 > 0$

∴ f is minimum at the point (0,0).

We have $z^2 = xy + 1$

$z^2 = 1$

$z = \pm 1$.

∴ points on the surface nearest to the origin is (0,0,1) & (0,0,−1)

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Q) Find the point on the surface $xyz^2=2$ nearest to the origin.

Sol: Let $O(0,0,0)$ be the origin.

Let $P(x,y,z)$ be any point on arbitary on the surface $xyz^2=2$.

$OP=\sqrt{x^2+y^2+z^2}$

$OP^2=x^2+y^2+z^2$

G.T, eq of the surface is $z^2=\frac{2}{xy}$ — ①

$OP^2=x^2+y^2+\frac{2}{xy}$ — ②

Let $f=x^2+y^2+\frac{2}{xy}$ — ③.

We have to minimize the function f satisfying the condition ①

Step-①: diff ③ w.r.t x & y partially.

$f_x=2x+\frac{2}{y}\left(\frac{-1}{x^2}\right)$

$f_y=2y-\frac{2}{xy^2}$.

Step-②: equate f_x & f_y to zero.

$2x-\frac{2}{x^2y}=0$ — ④

$2y-\frac{2}{xy^2}=0$ — ⑤

Step-③: Solving ④ & ⑤

$f_x=\frac{2x^3y-2}{yx^2}=0$, $f_y=\frac{y^3x-1}{xy^2}=0$.

$x^3y=1$ $\qquad$ $\therefore y^3x=1$

$x^3y=y^3x$.

$\Rightarrow x^3y-y^3x=0$.

$xy(x^2-y^2)=0$

$x^2-y^2=0 \Rightarrow x^2=y^2$ $\quad (\because x\neq0,\ y\neq0)$

$x=\pm y$

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$\therefore$ The stationary points are $P_1(1,1)$ $P_2(1,-1)$

$f_{xx} = 2 + \frac{4}{x^3 y}$, $f_{yy} = 2 + \frac{4}{y^3 x}$, $f_{xy} = \frac{2}{x^2 y^2}$.

At $(1,1)$ $r = 6 > 0$

$s = f_{xy} = 2$

$t = f_{yy} = 6$.

$rt - s^2 = 6(6) - 2 = 36 - 2 = 32 > 0$

Now $z^2 = \frac{2}{xy} \Rightarrow z = 2$

$\therefore z = \pm\sqrt{2}$

$\therefore$ The points on the surface nearest to the origin

is $(1, 1, \sqrt{2})$ $= \sqrt{1+1+2} = 2$.

$(1, 1, -\sqrt{2})$

Lagrange's method of undetermined multipliers:

Suppose $f(x, y, z)$ ① is a function of 3 variables x & y & z which are connected by the relation

$\phi(x, y, z) = 0$ — ②.

z value from ② can be solved & substituted in ①, the max or min of f can be found by $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ & testing $r > 0$ & $rt - s^2 > 0$ (or) $r < 0$

But in all cases it is not possible. We can use Lagrange's method.

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Working Procedure :

Suppose it is required to find the extremum for the function $f(x,y,z)$ subject to the condition $\phi(x,y,z)=0$ — ①

Step-①: Form the lagrangian function

$$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z) \quad — ②$$

where λ is called the lagrange multiplier which is determined by the following conditions.

Step-②: Diff ② w.r.t x,y & z partially & equate to zero,

$$\frac{\partial F}{\partial x}=0 \text{ i.e. } \frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \quad — ③$$

$$\frac{\partial F}{\partial y}=0 \text{ i.e } \frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \quad — ④$$

$$\frac{\partial F}{\partial z}=0 \text{ i.e } \frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \quad — ⑤.$$

Step-③: Solve the eq ①, ③, ④ & ⑤ the values of x,y,z so obtained will give the stationary point of $f(x,y,z)$.

Note:

To find the max or min for a function, $f(x,y,z)$ subject to the condition $\phi_1(x,y,z)=0$ & $\phi_2(x,y,z)=0$

Form the lagrangian function as

$$F(x,y,z)=f(x,y,z)+\lambda_1\phi_1(x,y,z)+\lambda_2\phi_2(x,y,z).$$

where λ_1 & λ_2 are lagrange multipliers.

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Q) Divide 24 into 3 points such that continued product of the first, square of the second & cube of third is maximum.

Sol: G.T, the number is 24.

Let x, y, z be the 3 parts of 24.

So, $x+y+z = 24$

G.T, the continued product of the first, square of the second & cube of the third i.e xy^2z^3

Let $f = xy^2z^3$

$$\phi = x+y+z-24 = 0 \quad -(1).$$

we have to maximize the function f and satisfies the condition (1).

Form the lagrangian function,

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$F = xy^2z^3 + \lambda(x+y+z-24) \quad -(2).$$

diff (2) w.r.t x, y, z partially

$$\frac{\partial F}{\partial x} = y^2z^3 + \lambda \quad -(3)$$

$$\frac{\partial F}{\partial y} = 2xyz^3 + \lambda \quad -(4)$$

$$\frac{\partial F}{\partial z} = 3xy^2z^2 + \lambda \quad -(5).$$

Equate $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ & $\frac{\partial F}{\partial z}$ to zero.

$\frac{\partial F}{\partial x} = 0$ i.e $y^2z^3 + \lambda = 0. \Rightarrow y^2z^3 = -\lambda \quad -(3).$

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$\frac{\partial F}{\partial y} = 0$ i.e $2xy^2z^3 + \lambda = 0 \Rightarrow 2xyz^3 = -\lambda$ — (4)

$\frac{\partial F}{\partial z} = 0$ i.e $3xy^2z^3 + \lambda = 0 \Rightarrow 3xy^2z^3 = -\lambda$ — (5).

$y^2z^3 = 2xyz^3 = 3xy^2z^2 = -\lambda$

Taking 1st two members, we get

$y^2z^3 = 2xyz^3$

$\Rightarrow y = 2x$ — (6)

Taking 2nd & 3rd members, we get

$2xyz^3 = 3xy^2z^2$

$z = \frac{3}{2}y$ — (7)

we have $x+y+z = 24$

$x + 2x + \frac{3}{2}y = 24$ [∵ from (6) & (7)]

$3x + \frac{3}{2}(2x) = 24$

$x = 4$

$y = 2(x)$

$y = 2(4) = 8$

$x+y+z = 24$

$4+8+z = 24$

$z = 24-12$

$z = 12$

∴ The stationary point is $(4, 8, 12)$

$f_{max} = xy^2z^3 = 4 \cdot 8^2 \cdot 12^3$

$= 442368.$

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Q) Sum of 3 numbers is constant. P.T their product is max when they are Equal.

Sol: G.T, the sum of 3 numbers is constant.

Let, $x+y+z=k$.

Let $f = xyz$.

$\phi = x+y+z-k = 0$. — ①

we have to minimize the function f and satisfies the condition ①-

From lagrangian function.

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$F = xyz + \lambda(x+y+z-k)$ — ②

diff ② w.r.t x, y, z partially.

$\frac{\partial F}{\partial x} = yz + \lambda$

$\frac{\partial F}{\partial y} = xz + \lambda$

$\frac{\partial F}{\partial z} = xy + \lambda$

Equate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ to zero.

$\frac{\partial F}{\partial x} = 0$ i.e $yz + \lambda = 0 \Rightarrow yz = -\lambda$ — ③.

$\frac{\partial F}{\partial y} = 0$ i.e $xz + \lambda = 0 \Rightarrow xz = -\lambda$ — ④

$\frac{\partial F}{\partial z} = 0$ i.e $xy + \lambda = 0 \Rightarrow xy = -\lambda$ — ⑤.

$xy = yz = zx = -\lambda$.

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Taking 1st two members, we get

$yz = zx$

$x = y$ — (6)

Taking 2nd & 3rd members, we get

$zx = xy$

$y = z$ — (7)

We have $x+y+z = k$

$y+y+y = k$

$3y = k$

$y = \frac{k}{3}$

$y = z$

$z = \frac{k}{3}$

$x = y$

$x = \frac{k}{3}$

$x+y+z = k$

$\frac{k}{3} + \frac{k}{3} + \frac{k}{3} = k$

∴ The stationary points is $\left(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}\right)$

$f_{max} = xyz = \frac{k^3}{3}$

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Q) Find a point on the plane $3x+2y+z=12$ which is nearest to the origin.

Sol: Let $O(0,0,0)$ be the Origin.

Let $P(x,y,z)$ be any point on the plane.

$OP = \sqrt{x^2+y^2+z^2}$

$OP^2 = x^2+y^2+z^2$

Let $f = x^2+y^2+z^2$

G.T, the eq of the plane $3x+2y+z=12$

Let $\phi = 3x+2y+z-12=0$ — ①

We have to minimiz the function f and subject to the Condition ① $\phi(x,y,z)=0$

Form the lagrangian function.

$F(x,y,z) = f(x,y,z)+\lambda\phi(x,y,z)$

$F = (x^2+y^2+z^2)+\lambda(3x+2y+z-12)$ — ②

diff ② w.r.t x,y,z partially

$\frac{\partial F}{\partial x} = 2x+3\lambda$

$\frac{\partial F}{\partial y} = 2y+2\lambda$

$\frac{\partial F}{\partial z} = 2z+\lambda$

Equate $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ & $\frac{\partial F}{\partial z}$ to zero.

$\frac{\partial F}{\partial x}=0$ i.e $2x+3\lambda=0 \Rightarrow$ $\frac{2x}{3} = -\lambda$ — ③

$\frac{\partial F}{\partial y}=0$ i.e $y+\lambda=0 \Rightarrow$ $y = -\lambda$ — ④

$\frac{\partial F}{\partial z}=0$ i.e $2z+\lambda=0 \Rightarrow$ $2z = -\lambda$ — ⑤

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From ③, ④ & ⑤ we can write

$$\frac{2x}{3} = y = 2z = -\lambda.$$

we have $3x+2y+z = 12$

$$3x + 2\left(\frac{2x}{3}\right) + \frac{x}{3} = 12$$

$$\left[\because 2z = \frac{2x}{3}, \quad z = \frac{x}{3}\right]$$

$$x = \frac{18}{7}.$$

$$y = \frac{2x}{3} = \frac{2}{3}\times\frac{18}{7} = \frac{12}{7}.$$

$$z = \frac{18}{21} = \frac{6}{7}.$$

$\therefore$ The stationary point is $\left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}\right)$

Hence $\left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}\right)$ is the point on the plane nearest to the origin.

Minimum Value of $OP = \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2}$

$=$

Q) Find the point on the plane $x+2y+3z=4$ that is closest to the origin.

Sol: Let $O(0,0,0)$ be the Origin.

Let $P(x,y,z)$ be any point on the plane.

$$OP = \sqrt{x^2+y^2+z^2}$$

$$OP^2 = x^2+y^2+z^2.$$

Let $f = x^2+y^2+z^2.$

G.T, the eq of plane $x+2y+3z=4.$

Let $\phi = x+2y+3z-4=0$ —①

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we have to minimize the function f & subject to the condition ① $\phi(x,y,z)=0$.

from the lagrangian function.

$$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$$

$$F=(x^2+y^2+z^2)+\lambda(x+2y+3z-4) \quad \text{—②}$$

diff ② w.r.t x, y, z partially.

$$\frac{\partial F}{\partial x}=2x+\lambda$$

$$\frac{\partial F}{\partial y}=2y+2\lambda$$

$$\frac{\partial F}{\partial z}=2z+3\lambda$$

Equate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ to zero.

$\frac{\partial F}{\partial x}=0$, i.e. $2x+\lambda=0 \Rightarrow 2x=-\lambda$ —③

$\frac{\partial F}{\partial y}=0$ i.e $2y+2\lambda=0 \Rightarrow y=-\lambda$ —④

$\frac{\partial F}{\partial z}=0$ i.e $2z+3\lambda=0 \Rightarrow \frac{2z}{3}=-\lambda$ —⑤

from ③, ④ & ⑤ we can write

$$2x=y=\frac{2z}{3}=-\lambda$$

we have $x+2y+3z=4$

$$2x+4x+3x=4$$

$$9x=4$$

$$x=\frac{4}{9}$$

$$y=2x=2\left(\frac{4}{9}\right)=\frac{8}{9}$$

$$z=3x=3\left(\frac{8}{9}\right)=\frac{8}{3}$$

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$\therefore$ The Stationary point is $\left(\frac{4}{9}, \frac{8}{9}, \frac{4}{3}\right)$

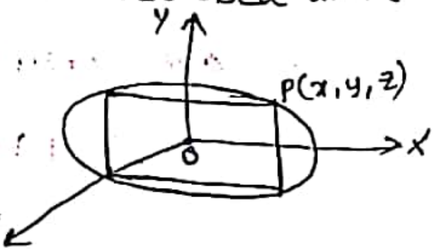
Hence $\left(\frac{4}{9}, \frac{8}{9}, \frac{4}{3}\right)$ is the point on the plane nearest to the origin.

Minimum Value of $OP = \sqrt{\frac{16}{81}+\frac{64}{81}+\frac{16}{9}}$

Q) Find the Volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol: G.T, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —① is the eq of ellipsoid

Let $2x, 2y, 2z$ be the length, breadth & height of the rectangular parallelopiped that can be inscribed in the ellipsoid.



Then the centroid of parallelopiped coincides with center O(0,0,0) of the ellipsoid & the corners of the parallelopiped lie on the surface of the ellipsoid ①.

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If (x, y, z) is any corner of the parallelopiped then it satisfies condition ①.

Let 'V' be the volume of parallelopiped

$V =$ i.e $2x \cdot 2y \cdot 2z = 8xyz$

Let $f = 8xyz$

We have to find the max value of 'V' i.e 'f'. subject to the condition ①.

Consider, the lagrangian function $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$.

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) \quad \text{—②}$$

diff ② w.r.t x, y, z partially.

$$\frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right)$$

$$\frac{\partial F}{\partial y} = 8xz + \lambda\left(\frac{2y}{b^2}\right)$$

$$\frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right).$$

Equate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ to '0'.

$\frac{\partial F}{\partial x} = 0$ i.e $8yz + \frac{2x}{a^2}\lambda = 0$.

$$\frac{a^2yz}{x} = \frac{-\lambda}{4} \quad \text{—③}$$

$\frac{\partial F}{\partial y} = 0$ i.e $8xz + \frac{2y\lambda}{b^2} = 0$

$$\frac{b^2xz}{y} = \frac{-\lambda}{4} \quad \text{—④}$$

$\frac{\partial F}{\partial z} = 0$ i.e $8xy + \frac{2z\lambda}{c^2} = 0$

$$\frac{c^2xy}{z} = \frac{-\lambda}{4} \quad \text{—⑤}.$$

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From ③, ④, & ⑤ we write

$$\frac{a^2yz}{x} = \frac{b^2xz}{y} = \frac{c^2xy}{z} = \frac{-\lambda}{4}$$

Taking 1st two members, we get

$$\frac{a^2yz}{x} = \frac{b^2xz}{y} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} \quad \text{— ⑥}$$

Taking 2nd & 3rd members, we get

$$\frac{b^2xz}{y} = \frac{c^2xy}{z} \Rightarrow \frac{y^2}{b^2} = \frac{z^2}{c^2} \quad \text{— ⑦}$$

From ⑥ & ⑦, we get:

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}. \quad \text{— ⑧}$$

we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1 \quad \{\because \text{From ⑧}\}$$

$$\frac{3x^2}{a^2} = 1$$

$$x = \pm\frac{a}{\sqrt{3}}$$

Similarly, $y = \pm\frac{b}{\sqrt{3}}$

$$z = \pm\frac{c}{\sqrt{3}}.$$

The stationary point is $\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{b}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}}\right)$

Maximum volume $V = 8xyz = 8\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}}$

$$= \frac{8abc}{3\sqrt{3}}.$$

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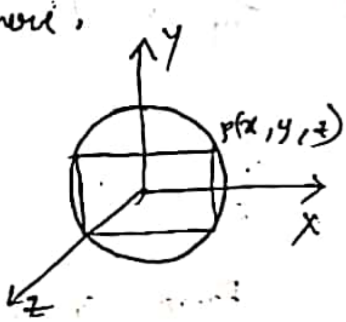
Q) Find the Volume of the greatest rectangular parallelopiped that can be inscribed in the sphere $x^2+y^2+z^2=a^2$

Sol: G.T, $x^2+y^2+z^2=a^2$ —① is the eq of Sphere.

Let $2x, 2y, 2z$ be the l, b & h of rectangular parallelopiped. that can be inscribed in sphere.



Then the Centroid of parallelopiped coincides with Center $O(0,0,0)$ of the sphere and corners of parallelopiped lie on the surface of sphere ①.

If (x,y,z) is any corner of parallelopiped then it satisfies condition ①

Let 'V' be the volume of parallelopiped

Let $f = 8xyz$

we have to find the max value of 'V' i.e 'f'.

Subject to condition ①.

Consider the lagrangian function $F(x,y,z) =$

$$f(x,y,z) + \lambda\phi(x,y,z).$$

$$F = 8xyz + \lambda(x^2+y^2+z^2-a^2) \quad —②$$

diff ② w.r.t x, y, z partially.

$$\frac{\partial F}{\partial x} = 8yz + \lambda(2x).$$

$$\frac{\partial F}{\partial y} = 8xz + \lambda(2y)$$

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$\frac{\partial F}{\partial z} = 8xy + \lambda(2z)$

Equate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ to zero.

$\frac{\partial F}{\partial x} = 0$ i.e $8yz + 2\lambda x = 0$.

$$\frac{4yz}{x} = -\lambda \quad \text{— (3)}$$

$\frac{\partial F}{\partial y} = 0$ i.e $8xz + 2\lambda y = 0$

$$\frac{4xz}{y} = -\lambda \quad \text{— (4)}$$

$\frac{\partial F}{\partial z} = 0$ i.e $\frac{4xy}{z} = -\lambda$ — (5)

From (3), (4) & (5) we get.

$$\frac{4yz}{x} = \frac{4xz}{y} = \frac{4xy}{z} = -\lambda$$

Taking 1st two members, we get

$$\frac{4y\cancel{z}}{x} = \frac{4x\cancel{z}}{y} \Rightarrow \frac{y}{x} = \frac{x}{y}$$

$$y^2 = x^2 \quad \text{— (6)}$$

Taking 2nd & 3rd members, we get.

$$\frac{4xz}{y} = \frac{4xy}{z} \Rightarrow \frac{z}{y} = \frac{y}{z}$$

$$z^2 = y^2 \quad \text{— (7)}$$

From (6) & (7)

$$x^2 = z^2 \quad \text{— (8)}$$

we have $x^2 + y^2 + z^2 = a^2$

$$x^2 + x^2 + x^2 = a^2$$

$$3x^2 = a^2$$

$$x^2 = \frac{a^2}{3}$$

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$x = \pm \frac{a}{\sqrt{3}}$

$|||^{ly}$, $y = \pm \frac{a}{\sqrt{3}}$

$z = \pm \frac{a}{\sqrt{3}}$

The Stationary point is $\left(\pm \frac{a}{\sqrt{3}}, \pm \frac{a}{\sqrt{3}}, \pm \frac{a}{\sqrt{3}}\right)$

Max volume $V = 8xyz = 8\left(\frac{a}{\sqrt{3}} \cdot \frac{a}{\sqrt{3}} \cdot \frac{a}{\sqrt{3}}\right)$

$$= \frac{8a^3}{3\sqrt{3}}.$$

8) Find extremum value of $x+y+z$ Subject to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Sol: This is a constrained extreme problem where the function $f(x,y,z) = x+y+z$ Subjected to the Constraint $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

So, consider the auxiliary function,

$$F(x,y,z) = x+y+z+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right) \quad \text{--- (1)}$$

Diff (1) w.r.t x, y, z partially & equate to '0'.

$$\frac{\partial F}{\partial x} = 1 - \frac{\lambda}{x^2} = 0 \quad \text{--- (2)}$$

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$$\frac{\partial F}{\partial y} = 1-\frac{\lambda}{y^2}=0 \quad \text{---} (3)$$

$$\frac{\partial F}{\partial z} = 1-\frac{\lambda}{z^2}=0 \quad \text{---} (4)$$

Solve (1), (3), (4) for x, y, z we get

$$x = \pm\sqrt{\lambda}$$

$$y = \pm\sqrt{\lambda}$$

$$z = \pm\sqrt{\lambda}$$

Sub these values of x, y, z in given constraint,

$$\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}=1$$

$$3=\sqrt{\lambda}$$

$$\lambda=9$$

using this λ we get $x=\pm 3,\ y=\pm 3,\ z=\pm 3$

Thus the max & min values are 9 & -9.

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Q) Find the max Value of $x^m y^n z^p$ when $x+y+z=a$.

Sol: G.T, $f = x^m y^n z^p$.

f subject to the Condition $x+y+z=a$ — ①

Consider the lagranges function

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$.

$F = x^m y^n z^p + \lambda(x+y+z-a)$ — ②

diff ② w.r.t x, y, z partially

$\frac{\partial F}{\partial x} = m x^{m-1} y^n z^p + \lambda$

$\frac{\partial F}{\partial y} = n y^{n-1} x^m z^p + \lambda$

$\frac{\partial F}{\partial z} = p z^{p-1} x^m y^n + \lambda$

Equate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ to '0'.

$m x^{m-1} y^n z^p + \lambda = 0$

$m x^{m-1} y^n z^p = -\lambda$ — ③.

$n y^{n-1} x^m z^p = -\lambda$ — ④

$p z^{p-1} x^m y^n = -\lambda$ — ⑤

from above eqns

$m x^{m-1} y^n z^p = n y^{n-1} x^m z^p = p z^{p-1} x^m y^n = -\lambda$.

Taking 1st two members, we get.

$m x^{m-1} y^n \cancel{z^p} = n y^{n-1} x^m \cancel{z^p}$

$my = nx$ — ⑥.

$\frac{x}{m} = \frac{y}{n}$

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Taking 2nd & 3rd members, we get.

$$ny^{n-1}x^m z^p = pz^{p-1}x^m y^n$$

$$nz = yp \quad \text{— (7)}$$

From (6) & (7)

$$px = zm. \quad \text{— (8)} \Rightarrow \frac{x}{m} = \frac{z}{p}.$$

we have $x+y+z=a$

$$x + \frac{nx}{m} + \frac{px}{m} = a$$

$$mx + nx + px = am$$

$$x(m+n+p) = am$$

$$x = \frac{am}{m+n+p}.$$

$$y = \frac{nx}{m} = \frac{n}{\cancel{m}}\left(\frac{a\cancel{m}}{m+n+p}\right)$$

$$z = \frac{px}{m} = \frac{p}{\cancel{m}}\left(\frac{a\cancel{m}}{m+n+p}\right)$$

The stationary point is $\left(\frac{am}{m+n+p}, \frac{na}{(m+n+p)}, \frac{pa}{(m+n+p)}\right)$

The max value of $x^m y^n z^p = \left(\frac{am}{m+n+p}\right)^m \left(\frac{na}{(m+n+p)}\right)^n \left(\frac{pa}{(m+n+p)}\right)^p$

$$= \frac{a^{m+n+p} \cdot n^n \cdot p^p \cdot m^m}{(m+n+p)^{m+n+p}}$$

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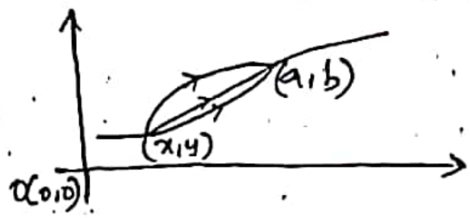
Limit of a function of two Variables :-

Let a function $f(x,y)$ we define in a region R. The function $f(x,y)$ is said tends to limit 'l' as $x \to a$, $y \to b$. If given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $|f(x,y) - l| < \varepsilon$.

whenever $|x-a| < \delta$, $|y-b| < \delta$.

Note: The limit exist if the value obtain is same along any path from (x,y) to (a,b) in X-Y-plane. i.e. lt as $x \to a$ and then $y \to b$ is equal to limit as $y \to b$ and then $x \to a$.



Continuity of a function of two Variables :-

A function $f(x,y)$ is said to be continuous at (a,b) if i) $\lim_{(x,y) \to (a,b)} f(x,y)$ exists

ii) $\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$

1) Examine for continuity at origin of the function defined by

$$f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & \text{for } (x,y) \neq (0,0) \\ \_\_\_ & \text{for } (x,y) = (0,0) \end{cases}$$

Redefined the function to make it continuous.

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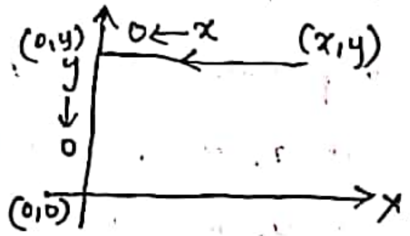
Sol.: The value of $f(x,y)$ for $x=0$ & $y=0$ is not given in the problem.

Continuity of a function at the point $(0,0)$

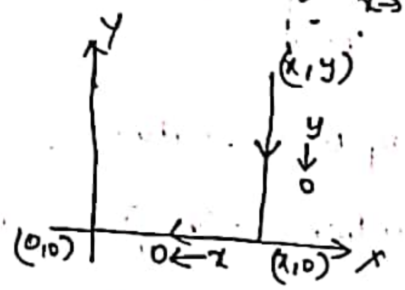
Case-①: As $x\to 0$ first and then $y\to 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{\sqrt{x^2+y^2}} = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+y^2}}\right] = 0.$$



Case-②: As $y\to 0$ first and then $x\to 0$.

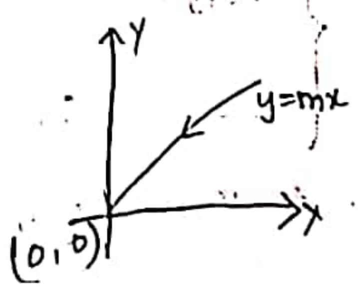
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{y\to 0\\ x\to 0}} \frac{x^2}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^2}{\sqrt{x^2+y^2}}\right] = \lim_{x\to 0} x = 0.$$



case-③: Along the line $y=mx$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{y=mx\\ x\to 0}} \frac{x^2}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x^2}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x\to 0} \frac{x}{\sqrt{1+m^2}} = 0.$$



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Hence the function $f(x,y)$ is continuous at the origin $(0,0)$. If $f(x,y)=0$ for $x=0, y=0$ otherwise $f(x,y)$ is not continuous at the origin.

The modified function is (for continuous)

$$f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

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Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum. 62

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Sol: Let x, y, z be three parts of 120.

$$x+y+z=120.$$

$$f = xy + yz + zx.$$

$$f = xy + y(120-x-y) + x(120-x-y).$$

$$f = \cancel{xy} + 120y - \cancel{xy} - y^2 + 120x - x^2 - yx$$

$$f = 120x + 120y - x^2 - y^2 - xy \quad \text{——} (1).$$

Diff (1) w.r.t 'x', y partially, we get

$$\frac{\partial f}{\partial x} = 120 - 2x - y.$$

$$\frac{\partial f}{\partial y} = 120 - 2y - x.$$

Equate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ to zero, we get

$\frac{\partial f}{\partial x} = 0$ i.e $120 - 2x - y = 0$ —— (2), $\frac{\partial f}{\partial y} = 0$ i.e $120 - 2y - x = 0.$ —— (3).

solving (2) and (3), we get

$$x = 40, \; y = 40.$$

The stationary point is $(40, 40)$.

$$l = \frac{\partial^2 f}{\partial x^2} = -2, \quad m = \frac{\partial^2 f}{\partial x \partial y} = -1, \quad n = \frac{\partial^2 f}{\partial y^2} = -2.$$

At the point $(40, 40)$

$$ln - m^2 = 4 - 1 = 3 > 0.$$

$$l = -2 < 0.$$

$\therefore$ f is maximum at $(40, 40)$

$\because x+y+z=120$

$z = 120-80$

$z = 40$

$$\therefore f_{max} = 40.40 + 40.40 + 40.40$$

$$f_{max} = 3.40.40$$

$$f_{max} = 4800$$

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Find the shortest distance from the point $(1,0)$ to the parabola $y^2 = 4x$.

Sol:- Given that the equation of the parabola $y^2=4x$

Let (x,y) be any point on the parabola $y^2=4x$.

The distance from $(1,0)$ to any point (x,y) on $y^2=4x$ is

$$p^2 = (x-1)^2+y^2.$$

$$\text{Let } f(x,y) = (x-1)^2+y^2 \quad \text{——} \ (1)$$

$$\phi(x,y) = y^2-4x \quad \text{——} \ (2)$$

Consider the Lagrangian function.

$$F = f(x,y) + \lambda\, \phi(x,y)$$

$$F = (x-1)^2+y^2+\lambda(y^2-4x) \quad \text{——} \ (3)$$

Diff (3) w.r.t x and y partially, we get.

$$\frac{\partial F}{\partial x} = 2(x-1)-4\lambda$$

$$\frac{\partial F}{\partial y} = 2y+2\lambda y$$

Equate $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ to zero, we get

$$\frac{\partial F}{\partial x}=0 \quad \text{i.e} \quad 2(x-1)-4\lambda=0 \quad \text{——} \ (4)$$

$$\frac{\partial F}{\partial y}=0 \quad \text{i.e} \quad 2y+2\lambda y=0 \quad \text{——} \ (5)$$

From (5), $y=0$ or $\lambda=-1$.

If $\lambda=-1$, we get $x-1=-2$ $[\because$ from (4)$]$

$x=-1$.

$\therefore$ $(-1,0)$ does not satisfy (2).

If $y=0$, $x=0$ from (2).

$\therefore$ $f(0,0)=1$.

$\therefore$ shortest distance is 1.

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As the dimensions of a triangle ABC are varied, show that the maximum values of $\cos A \cos B \cos C$ is obtained when the triangle is equilateral. 61

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Sol: Let $f(A,B,C) = \cos A \cos B \cos C$.

In a triangle ABC, $A+B+C = 180°$

$\phi(A,B,C) = A+B+C-180°$

Consider the Lagrangean function $F = f(A,B,C) + \lambda\, \phi(A,B,C)$

$$F = \cos A \cos B \cos C + \lambda (A+B+C-180°) \quad \text{--- (1)}$$

Diff (1) w.r.t A, B, C partially, we get

$$\frac{\partial F}{\partial A} = -\sin A \cos B \cos C + \lambda$$

$$\frac{\partial F}{\partial B} = -\cos A \sin B \cos C + \lambda$$

$$\frac{\partial F}{\partial C} = -\cos A \cos B \sin C + \lambda.$$

Equate. $\frac{\partial F}{\partial A}$, $\frac{\partial F}{\partial B}$ and $\frac{\partial F}{\partial C}$ to zero, we get

$\frac{\partial F}{\partial A} = 0$ i.e $-\sin A \cos B \cos C + \lambda = 0 \Rightarrow \lambda = \sin A \cos B \cos C$ --- (2)

$\frac{\partial F}{\partial B} = 0$ i.e $-\cos A \sin B \cos C + \lambda = 0 \Rightarrow \lambda = \cos A \sin B \cos C$ --- (3)

$\frac{\partial F}{\partial C} = 0$ i.e $-\cos A \cos B \sin C + \lambda = 0 \Rightarrow \lambda = \cos A \cos B \sin C$ --- (4)

From equations (2), (3) and (4).

$$\sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C.$$

Dividing by $\cos A \cos B \cos C$.

$$\tan A = \tan B = \tan C.$$

$$A = B = C.$$

∴ Hence, the triangle ABC is equilateral.

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Find the minimum value of $x^2+y^2+z^2$ with the constraint $xy+yz+zx=3a^2$.

Sol: Let $f = x^2+y^2+z^2$, $\phi = xy+yz+zx-3a^2$.

Consider the Lagrangean function $F = f(x,y,z)+\lambda\,\phi(x,y,z)$

$$F = (x^2+y^2+z^2)+\lambda(xy+yz+zx-3a^2) \text{ --- } (1).$$

Diff (1) w.r.t x, y and z partially, we get

$$\frac{\partial F}{\partial x} = 2x+\lambda(y+z)$$

$$\frac{\partial F}{\partial y} = 2y+\lambda(x+z)$$

$$\frac{\partial F}{\partial z} = 2z+\lambda(x+y)$$

Equate $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ to zero, we get

$\frac{\partial F}{\partial x}=0$ i.e $2x+\lambda(y+z)=0 \Rightarrow -\lambda = \frac{2x}{y+z}$ i.e $-\frac{1}{\lambda} = \frac{y+z}{2x}$. — (2)

$\frac{\partial F}{\partial y}=0$ i.e $2y+\lambda(x+z)=0 \Rightarrow -\lambda = \frac{2y}{x+z}$ i.e $-\frac{1}{\lambda} = \frac{x+z}{2y}$. — (3)

$\frac{\partial F}{\partial z}=0$ i.e $2z+\lambda(x+y)=0 \Rightarrow -\lambda = \frac{2z}{x+y}$ i.e $-\frac{1}{\lambda} = \frac{x+y}{2z}$. — (4).

From equations (2), (3) and (4), we get

$$\frac{y+z}{2x} = \frac{x+z}{2y} = \frac{x+y}{2z} = \frac{y+z+x+z+x+y}{2x+2y+2z} = 1.$$

$$\frac{y+z}{2x}=1 \Rightarrow 2x-y-z=0.$$

$$\frac{x+z}{2y}=1 \Rightarrow -x+2y-z=0.$$

$$\frac{x+y}{2z}=1 \Rightarrow -x-y-2z=0.$$

solving these equations, we get $x=y=z$.

sub. $y=z=x$ in $xy+yz+zx=3a^2$, we get

$$3x^2=3a^2 \Rightarrow x=\pm a.$$

$$x=y=z=\pm a.$$

$\therefore$ Minimum value of $f=x^2+y^2+z^2$ is $3a^2$.

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Show that if the perimeter of a triangle is a constant, the triangle has maximum area when it is equilateral.

Sol:- Let x, y and z be the sides of the triangle.

Perimeter of the triangle $s = \frac{x+y+z}{2}$

Area of the triangle $A = \sqrt{s(s-x)(s-y)(s-z)}$

$$\text{Let } f(x,y,z) = A^2 = s(s-x)(s-y)(s-z) \quad \text{--- (1)}$$

$$x+y+z = 2s.$$

$$\text{Let } \phi(x,y,z) = x+y+z-2s. \quad \text{--- (2)}$$

Consider the Lagrangian function $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$$F(x,y,z) = s(s-x)(s-y)(s-z) + \lambda(x+y+z-2s) \quad \text{--- (3)}$$

Diff (3) w.r.t x, y and z partially, and equate to zero, we get

$\frac{\partial F}{\partial x} = 0$ i.e $-s(s-y)(s-z)+\lambda = 0.$

$$\lambda = s(s-y)(s-z) \quad \text{--- (4)}$$

$\frac{\partial F}{\partial y} = 0$ i.e $-s(s-x)(s-z)+\lambda = 0$

$$\lambda = s(s-x)(s-z) \quad \text{--- (5)}$$

$\frac{\partial F}{\partial z} = 0$ i.e $-s(s-x)(s-y)+\lambda = 0$

$$\lambda = s(s-x)(s-y) \quad \text{--- (6)}$$

From eqn's (4), (5) and (6), we get

$$s(s-y)(s-z) = s(s-x)(s-z) = s(s-x)(s-y)$$

Taking 1st two members, we get

$$s(s-y)(s-z) = s(s-x)(s-z)$$

$$s-y = s-x$$

$$x = y \quad \text{--- (7)}$$

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Taking 2nd and 3rd members, we get

$$s(s-x)(s-z) = s(s-x)(s-y)$$

$$s-z = s-y$$

$$y = z \quad \text{—— (8)}$$

From (7) and (8), we get

$$x = y = z$$

$\therefore$ The triangle is equilateral.

A wire of length b is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found.

Sol:- Given that the length of the wire is b.

Let x and y be the two parts (pieces) of wire. $(x+y=b)$

Let the piece of length x be bent in the form of a square so that each side is $\frac{x}{4}$.

The area of the square $A_1 = \frac{x}{4}\cdot\frac{x}{4} = \frac{x^2}{16}$.

Suppose a piece of length y is bent in the form of a circle of radius r

so perimeter of the circle is y.

$$2\pi r = y$$

$$r = \frac{y}{2\pi}$$

The area of the circle $A_2 = \pi\left(\frac{y}{2\pi}\right)^2 = \frac{y^2}{4\pi}$

Let sum of the areas be given as

$$f(x,y) = A_1 + A_2$$

$$f(x,y) = \frac{x^2}{16} + \frac{y^2}{4\pi} \quad \text{—— (1)}$$

Also $\quad x+y = b \quad \text{—— (2)}$

Let $\phi(x,y) = x+y-b$.

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Consider the Lagrangian function $F(x,y,\lambda) = f(x,y) + \lambda\,\phi(x,y)$

$$F(x,y) = \left(\frac{x^2}{16} + \frac{y^2}{4\pi}\right) + \lambda(x+y-b) \quad \text{—— (3)}$$

Diff (3) w.r.t x and y partially, we get

$$\frac{\partial F}{\partial x} = \frac{x}{8} + \lambda$$

$$\frac{\partial F}{\partial y} = \frac{y}{2\pi} + \lambda$$

Equate $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ to zero, we get

$$\frac{\partial F}{\partial x} = 0 \quad \text{i.e} \quad \frac{x}{8} + \lambda = 0 \implies \frac{x}{8} = -\lambda \quad \text{—— (4)}$$

$$\frac{\partial F}{\partial y} = 0 \quad \text{i.e} \quad \frac{y}{2\pi} + \lambda = 0 \implies \frac{y}{2\pi} = -\lambda \quad \text{—— (5)}$$

From (4) and (5), we get

$$\frac{x}{8} = \frac{y}{2\pi}$$

$$x = \frac{4}{\pi}y.$$

We have $x + y = b$

$$\implies \frac{4}{\pi}y + y = b$$

$$y = \frac{b\pi}{4+\pi}$$

$$x = b - y = b - \frac{b\pi}{4+\pi}$$

$$x = \frac{4b}{4+\pi}$$

$\therefore$ The stationary pt is $\left(\frac{4b}{4+\pi}, \frac{b\pi}{4+\pi}\right)$

$\therefore$ The least value of the sum of the areas is

$$f = \frac{x^2}{16} + \frac{y^2}{4\pi} = \frac{1}{16}\left(\frac{4b}{4+\pi}\right)^2 + \frac{1}{4\pi}\left(\frac{\pi b}{4+\pi}\right)^2$$

$$f = \frac{b^2}{4(4+\pi)}$$

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Taylor's series for a function of two variables :-

If $f(x,y)$ possess continuous partial derivatives of n^{th} order in any neighbourhood of a point (x,y) and if $(x+h, y+k)$ is any point of this neighbourhood, then .

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y)$$
$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x,y) + \cdots$$

Note (i) :- $f(a+h, b+k) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] +$
$$\left[\frac{h^2}{2!} f_{xx}(a,b) + hk f_{xy}(a,b) + \frac{k^2}{2!} f_{yy}(a,b)\right] + \cdots$$

(ii) Put $a+h = x \Rightarrow h = x-a$ and $b+k = y \Rightarrow k = y-b$.

From Note (i) .

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) +$$
$$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

(iii) Put $a=b=0$, $h=x$, $k=y$ in the above.

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + f_{yy}(0,0)\right] + \cdots$$

This is known as Maclaurin's series for two variables.

→ Expand e^{xy} in the neighbourhood of $(1,1)$.

Sol:- Let $f(x,y) = e^{xy}$

We have to expand $f(x,y)$ in the neighbourhood of $(1,1)$.

The Taylor's series expansion of $f(x,y)$ about (a,b) is given by.

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Here $a=1$, $b=1$.

$$f(x,y) = f(1,1) + (x-1)f_x(1,1) + (y-1)f_y(1,1) + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \cdots \quad (1)$$

$f(x,y) = e^{xy}$ $\qquad$ $f(1,1) = e$.

$f_x(x,y) = ye^{xy}$ $\qquad$ $f_x(1,1) = e$.

$f_{xx}(x,y) = y^2 e^{xy}$ $\qquad$ $f_{xx}(1,1) = e$.

$f_y(x,y) = xe^{xy}$ $\qquad$ $f_y(1,1) = e$.

$f_{yy}(x,y) = x^2 e^{xy}$ $\qquad$ $f_{yy}(1,1) = e$.

$f_{xy}(x,y) = xy\,e^{xy} + e^{xy}$ $\qquad$ $f_{xy}(1,1) = 2e$.

Sub. all these in (1), we get

$$e^{xy} = e + e(x-1) + e(y-1) + \frac{1}{2!}\left[e(x-1)^2 + 4e(x-1)(y-1) + e(y-1)^2\right] + \cdots$$

$$e^{xy} = e\left[1 + (x-1) + (y-1) + \frac{(x-1)^2}{2!} + 2(x-1)(y-1) + \frac{(y-1)^2}{2!} + \cdots\right].$$

MODULE -IV

ORDINARY DIFFERENTIAL EQUATIONS

Differential Equations of first order and their applications

Ordinary Differential Equations of First order and First Degree.

Differential Equation :—

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called differential equation.

Types of Differential Equations :—

(a) Ordinary Differential Equation :-

A differential equation is said to be ordinary if the derivatives in the equation have reference to only a single independent variable.

Eg:-

$$\left(\frac{dy}{dx}\right)^3 - 5\left(\frac{dy}{dx}\right)^2 + 6y = \sin x$$

$$\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^3 - 6y = \log x$$

$$(x^2+y^2-x)dy + (ye^y - 2xy)dx = 0.$$

(b) Partial Differential Equation :—

A differential equation is said to be partial if the derivatives in the equation have reference to two or more independent variables.

Eg:- $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

The order of a Differential Equation :—

The order of a differential equation is the order of the highest derivative appearing in the equation.

Eg:- (a) $(x^2+1)\frac{dy}{dx} + 2xy = 4x^2$

The first derivative $\frac{dy}{dx}$ is the highest derivative in the above equation

$\therefore$ The order of the above equation is 1.

(b) $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$

The second derivative $\frac{d^2y}{dx^2}$ is the highest derivative in the above equation.

$\therefore$ The order of the above equation is 2.

The Degree of a Differential Equation :—

The degree of a differential equation is the degree of that highest derivative when the derivatives are free from radicals and fractions.

(OR)

Let $F(x, y, y', y'', \ldots y^{(n)}) = 0$ be a differential equation of order n.
If the given differential equation is a polynomial in $y^{(n)}$, then the highest degree of $y^{(n)}$ is defined as the degree of the differential equation.

Note:- (1) If in the given equation $y^{(n)}$ enters in the denominator or has a fractional index, then it may be possible to free it from radicals by algebraic operations so that $y^{(n)}$ has the least positive integral Index and the equation is written as a polynomial in $y^{(n)}$.

(2) The above definition of degree does not require variables x, t, u etc to be free from radicals and fractions.

(3) If it is not possible to express the differential equation as a polynomial in $f^{(n)}$, then the degree of the differential equation is not defined.

Eg:- (a) $y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\left(y - x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + (1-y^2) = 0.$$

This is a polynomial equation in $\frac{dy}{dx}$.

The highest degree of $\frac{dy}{dx}$ is two.

Hence the degree of the above differential equation is 2.

1 b) $a\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

$$a^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

This is a polynomial equation in $\frac{d^2y}{dx^2}$

The highest degree of $\frac{d^2y}{dx^2}$ is 2

Hence the degree of the above differential equation is 2.

(c) $y = \cos\left(\frac{dy}{dx}\right)$ and $x = y + \log\left(\frac{dy}{dx}\right)$

The above equations can not be expressed as polynomial equations in $\frac{dy}{dx}$.

Hence the degree of the above differential equations can not be determined and hence undefined.

		Order	Degree
1	$\frac{dy}{dx} = e^x$	1	1
2	$\left(\frac{dy}{dx}\right)^2 = ax^2 + bx + c$	1	2
3	$\left(\frac{d^3y}{dx^3}\right)^3 = x^2 \frac{dy}{dx}$	3	3
4	$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 2y = 0$	2	1
5	$\frac{dy}{dx} = \frac{x+y}{\frac{dy}{dx}}$	1	2
6	$y = xy'' + x\sqrt{1+(y')^2}$	2	2
7	$y = x\frac{dy}{dx}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$	1	4
8	$y = \sin\left(\frac{dy}{dx}\right)$	1	Not defined
9	$\frac{d^2y}{dx^2} + \left[1+\left(\frac{dy}{dx}\right)^4\right] = 0$	2	1
10	$y'' + x^2y' + 2xy^2 = \sin x$	2	1

Solution of a differential equation :—

Any relation between the dependent and independent variables not containing their derivatives which satisfies the given diff. eqn is called a solution or integral of the diff. eqn.

Eg:- $y = a\cos x + b\sin x$ is a sol. of $\frac{d^2y}{dx^2} + y = 0$.

Observe that $y = a\cos x + b\sin x$ is a sol. of the given diff eqn for any real constants a and b which are called arbitrary constants.

General Solution :-

A solution containing the number of independent arbitrary constants which is equal to the order of the diff. eqn is called the general solution or complete primitive of the equation.

Eg:- $y = c_1 e^x + c_2 e^{2x}$ is the general solution of $y'' - 3y' + 2y = 0$, as it contains two independent arbitrary constants.

Particular Solution :—

A solution obtained from the general solution of a diff. equation by giving particular values to the independent arbitrary constants is called a particular solution to the given diff. eqn.

Eg:- Some particular solutions of $y'' - 3y' + 2y = 0$ is given by $y = e^x + e^{2x}$, $y = e^x - 2e^{2x}$ etc.

Singular solution :—

A solution which can not be obtained from any general solution of a diff. equation by any choice of the independent arbitrary constants is called a singular solution of the given diff. equation.

Eg:- $y = (x+c)^2$ —(1) is the general solution of $y_1^2 - 4y = 0$ —(2).

$y = 0$ is also a solution of (2). More over $y = 0$ can not be obtained by any choice of c in (1)

Hence $y = 0$ is a singular solution of (2).

Orthogonal Trajectories :—

Trajectory :- A curve that intersects each member of a family of curves according to some specified property is called Trajectory of the family of curves.

Orthogonal trajectory :- A trajectory which cuts every member of a family of curves at right angles is called an orthogonal traje-ctory of the given family of curves.

Orthogonal Trajectories in cartesian form :—

Let the family of curves be described by the equation

$$f(x,y,c)=0 \quad \text{——— (1)}$$

Where c is a parameter.

Diff. (1) we have.

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0.$$

$$\frac{dy}{dx}=\frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{——— (2)}.$$

Eliminating c between (1) and (2), we get

$$F\left(x,y,\frac{dy}{dx}\right)=0 \quad \text{——— (3)}.$$

Equation (3) represents the diff. eqn of the family of curves given by (1).

If the slope of any member of (1) at (x,y) on the curve is $\frac{dy}{dx}$, then the slope of the curve passing through the.

point (x,y) and cutting the member curve orthogonally is $\left(-\frac{dx}{dy}\right)$

$\therefore$ The slope of a member of the members of family of ortho-
-gonal trajectories of ① is $-\left(\frac{dx}{dy}\right)$

Hence the diff. equation of the orthogonal trajectories may be obtained by replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

The orthogonal trajectories of ① can be obtained by solving

$$F\left(x,y,-\frac{dx}{dy}\right)=0.$$

<u>Working Procedure</u> :—

<u>Step 1</u> :- Let the cartesian equation of the family of curves be $f(x,y,c)=0$ —— ①

<u>Step 2</u> :- Diff ① w.r.t x and eliminate c, we get the differential equation of the family of curves be

$$F\left(x,y,\frac{dy}{dx}\right)=0 \quad \text{——} ②.$$

<u>Step 3</u> :- Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in ②, we get the differen-
-tial equation of the family of orthogonal trajectories.

$$F\left(x,y,-\frac{dx}{dy}\right)=0 \text{ —— } ③$$

<u>Step 4</u> :- Solving the diff. eqn ③ to get the orthogonal trajectory.

→ Show that the system of confocal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, where λ is a parameter is self orthogonal.

Sol: Given that the equation of the family of confocal conics is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \quad \text{——(1)}$$

Diff (1) w.r.t "x", we get

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\cdot\frac{dy}{dx}=0.$$

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\cdot p=0 \quad \text{where } p=\frac{dy}{dx}.$$

$$x(b^2+\lambda)+y(a^2+\lambda)p=0.$$

$$(xb^2+ya^2p)+\lambda(x+yp)=0.$$

$$\lambda=\frac{-(b^2x+a^2yp)}{x+yp}$$

$$\left.\begin{aligned} \therefore\ a^2+\lambda &= a^2-\frac{(b^2x+a^2yp)}{x+yp}=\frac{(a^2-b^2)x}{x+yp} \\ b^2+\lambda &= b^2-\frac{(b^2x+a^2yp)}{x+yp}=\frac{-(a^2-b^2)yp}{x+yp} \end{aligned}\right\} \text{——(2)}.$$

Eliminating λ from (1) and (2), we get

$$\frac{x^{\cancel{2}}(x+yp)}{(a^2-b^2)\cancel{x}}+\frac{y^{\cancel{2}}(x+yp)}{(a^2-b^2)\cancel{y}p}=1$$

$$\frac{x+yp}{a^2-b^2}\left(x-\frac{y}{p}\right)=1.$$

$$(x+yp)\left(x-\frac{y}{p}\right)=a^2-b^2 \quad \text{——(3)}$$

This is differential equation of family of curves (1).

We get the differential equation of the family of orthogonal trajectories.

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by repacing $\frac{dy}{dx} = p$ with $-\frac{dx}{dy} = -\frac{1}{\frac{dy}{dx}} = \frac{-1}{p}$.

Hence the differential equation of orthogonal trajectories is.

$$\left(x - \frac{y}{p}\right)(x + py) = a^2 - b^2 \quad \text{——} \quad (4).$$

Which is same as (3). Thus we see that the differential equation of the family of orthogonal trajectories is same as that of the orthogonal family.

Hence the given family of curves is orthogonal to itself.

Hence it is a self orthogonal family of curves.

ORTHOGONAL TRAJECTORIES (Cartesian)

1 Find the orthogonal trajectories of the family of curves $y = \frac{x}{1+c_1 x}$ where c_1 is the parameter. Ans:- $x^3 + y^3 = c_2$.

2 Find the orthogonal trajectories of the family of parabolas through the origin and the foci on y-axis. Ans:- $\frac{x^2}{2} + \frac{y^2}{1} = c$.

3 Find the orthogonal trajectories of the family of curves $4y + x^2 + 1 + c_1 e^{2y} = 0$ where c_1 is the parameter. A:- $y = \frac{1}{4} - \frac{1}{6}x^2 + c_2 x^{-4}$.

4 Find the member of the O.T for the curve $x+y = ce^y$ which passes through $(0,5)$. Ans:- $ye^x = 2e^x - xe^x + c_2$, $y = 2 - x + 3e^{-x}$.

5 Find the O.T of the family of co axial circles $x^2+y^2+2gx+c=0$ where g is parameter. Ans:- $x^2+y^2-c_1 y + c = 0$.

6 Find the O.T of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$ where λ is the parameter. Ans:- $x^2+y^2-2a^2\log x = c$.

7 Show that the family of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self orthogonal. Where λ is the parameter.

8 Show that the orthogonal trajectories of the system of parabolas $y^2 = 4a(x+a)$ belongs to the system itself, 'a' being the parameter.

9 Find the family orthogonal to the family $y = ce^{-x}$ is of exponential curves. Determine the member of each family passing through $(0,4)$.

Ans:- $y = 4e^{-x}$, $y^2 = 2(x+8)$

10 Find the O.T of the family $y = x + ce^{-x}$ and determine the particular member of each family that passes through $(0,3)$.

Ans:- $y = x + 3e^{-x}$, $x - y + 2 + e^{3-y} = 0$.

11 Find the O.T of the family of curves whose diff. equation is

$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$ Ans:- $x^2+y^2 = cy$.

12 Find the O.T of the family of curves $x^2+y^2+2gx+c=0$ where g is the parameter. Ans:- $x^2+y^2-ky-c=0$.

13 show that the family of parabolas $x^2=4a(a+y)$ is self orthogonal where a is parameter.

14 show that the family of confocal conics $\frac{x^2}{a^2}+\frac{y^2}{a-b}=1$ is self orthogonal. Here a is the parameter and b is the constant.

15 Find the value of the constant d such that the parabolas $y=c_1x^2+d$ are the orthogonal trajectories of the family of ellipses $x^2+2y^2-y=c_2$

Ans:- $d=\frac{1}{4}$.

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Polar Coordinates :-

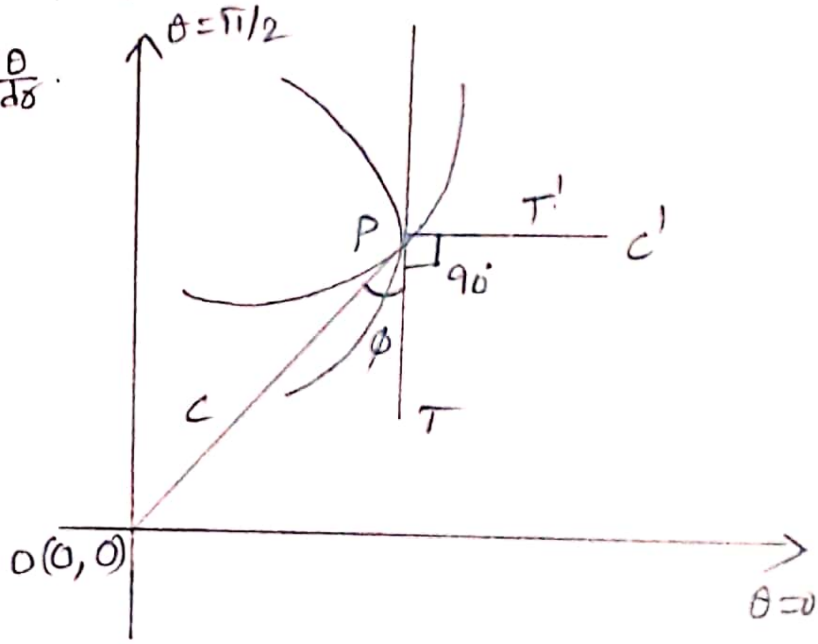
Suppose $f(r, \theta, k) = 0$ is the given family of curves. Forming the differential equation eliminating the arbitrary constant.

The differential equation is $F(r, \theta, \frac{dr}{d\theta}) = 0$.

Suppose c is curve in the family $f(r, \theta, k) = 0$.

Suppose c is any curve which cuts c' orthogonally

Let PT be the tangent to c at P and PT' is the tangent to c' at P. $\angle OPT = \phi$, $\angle OPT' = 90° + \phi = \phi'$



We know that $\tan\phi = r\frac{d\theta}{dr}$.

$$\tan\phi' = \tan(90+\phi)$$
$$= -\cot\phi$$
$$= -\frac{1}{\tan\phi}$$
$$= \frac{-1}{r\frac{d\theta}{dr}}$$
$$= -\frac{1}{r}\frac{dr}{d\theta}$$

On replacing $\frac{d\theta}{dr}$ by $-\frac{1}{r}\frac{dr}{d\theta}$ we get the differential equation of the orthogonal trajectories.

$\frac{dr}{d\theta}$ should be replaced by $-r^2\frac{d\theta}{dr}$

Solving the differential equation, we get the family of orthogonal trajectories.

Orthogonal Trajectories in Polar form :-

Working Procedure :-

Let $f(r, \theta, c) = 0$ —① be the equation of the given family of curves in polar form

Step 1 :- Diff ① w.r.t 'θ' and obtain the differential equation of family of curves $F(r, \theta, \frac{dr}{d\theta}) = 0$ —② by eliminating the parameter c.

Step 2 :- Replace $\frac{dr}{d\theta}$ with $-r^2 \frac{d\theta}{dr}$ in ②.

Then the differential equation of the family of orthogonal trajectories $F(r, \theta, -r^2 \frac{d\theta}{dr}) = 0$ —③

Step 3 :- Solve the equation ③ to get the equation of the orthogonal trajectories of ① .

→ Find the orthogonal trajectories of the family of cardioids $r = a(1-\cos\theta)$ where a is the parameter.

Sol:- Given equation of the family of cardioids is $r = a(1-\cos\theta)$ —— ①

Diff ① w.r.t θ, we get

$$\frac{dr}{d\theta} = a\sin\theta$$

$$a = \frac{1}{\sin\theta}\frac{dr}{d\theta} \quad \text{—— ②}$$

Eliminating a from ① and ②, we get

$$r = \frac{1}{\sin\theta}(1-\cos\theta)\frac{dr}{d\theta}$$

$$\frac{dr}{d\theta} = \frac{r\,\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\sin^{\cancel{2}}\theta/2}$$

$$\frac{dr}{d\theta} = r\cot\theta/2 \quad \text{—— ③}$$

This is the differential equation of family of given curves

To get differential equation of family of orthogonal trajectories replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$ in ③, we get

$$-r^{\cancel{2}}\frac{d\theta}{dr} = -\cancel{r}\cot\theta/2$$

$$-r\frac{d\theta}{dr} = \cot\theta/2$$

Separating the variables and integrate both sides, we get

$$\int\frac{-d\theta}{\cot\theta/2} = \int\frac{dr}{r} + \log c$$

$$\int -\frac{\sin\theta/2}{\cos\theta/2} = \log|r| + \log c$$

$$2\int\frac{-\frac{1}{2}\sin\theta/2}{\cos\theta/2} = \log(cr)$$

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$$2\log(\cos\theta/2) = \log(c\gamma)$$

$$\log\cos^2\theta/2 = \log(c\gamma)$$

$$\cos^2\theta/2 = c\gamma$$

$$c\gamma = \frac{1+\cos\theta}{2}$$

$$\gamma = \frac{1}{2c}(1+\cos\theta)$$

$$\gamma = c_1(1+\cos\theta)$$

This is the equation of family of orthogonal trajectories.

ORTHOGONAL TRAJECTORIES (Polar form)

1 Find the orthogonal trajectories of the family of curves $r = a\theta$ where a is the parameter Ans:- $r = c\,e^{-\theta^2/2}$.

2 Find an eqn of the O.T of the family of circles having polar equation $r = 2a\cos\theta$ where a is the parameter Ans:- $r = 2c\sin\theta$.

3 Find an O.T of the family of curves $r^2 = a^2\cos 2\theta$ where a is the parameter Ans:- $r^2 = c^2\sin 2\theta$.

4 Find an O.T of the family of curves $r^n\sin(n\theta) = a^n$ where a is the parameter Ans:- $r^n\cos n\theta = c_1^n$

5 Find an O.T of the family of curves $r = \frac{2a}{1+\cos\theta}$ where a is the parameter Ans:- $r = \frac{2c}{1-\cos\theta}$.

6 Find an O.T of the family of curves $r = a(1+\cos\theta)$ where a is the parameter Ans:- $r = c(1-\cos\theta)$.

7 P/T the orthogonal trajectories of the family of curves $A = r^2\cos\theta$ are the curves $B = r\sin^2\theta$ where A and B are parameter.

8 Find the O.T of $r = a(1-\sin\theta)$ where a is the parameter Ans:- $r = c(1+\sin\theta)$.

9 Find the O.T of $r = a(1-\cos\theta)$ where a is the parameter.

Newton's Law of Cooling :—

Statement :- The rate of change of the temparature of a body is proportional to the difference of the temparature of the body and that of the surrounding medium.

Let θ be the temparature of the body at time t and θ_0 be the temparature of its surrounding medium (usually air).

By the Newton's Law of cooling, we have.

$$\frac{d\theta}{dt} \propto (\theta-\theta_0)$$

$$\frac{d\theta}{dt} = -k(\theta-\theta_0)$$ where k is a +ve constant

Separate the variables and integrate, we get

$$\int \frac{d\theta}{\theta-\theta_0} = -k\int dt + \log c.$$

$$\log|\theta-\theta_0| = -kt + \log c.$$

$$\log|\theta-\theta_0| - \log c = -kt$$

$$\log\left|\frac{\theta-\theta_0}{c}\right| = -kt$$

$$\frac{\theta-\theta_0}{c} = e^{-kt}$$

$$\theta = \theta_0 + c\, e^{-kt} \quad \text{—} (1)$$

If initially $\theta = \theta_1$ is the temparature of the body at time $t=0$

Then (1) gives, $c = \theta_1 - \theta_0$ — (2)

Sub (2) in (1), we get

$$\theta = \theta_0 + (\theta_1-\theta_0)e^{-kt}$$

11) A body is originally at 80°c and cools down to 60°c in 20min. If the temperature of the air is 40°c find the temperature of the body after 40 minutes.

Sol:- Let θ be the temperature of the body at time t

The temperature of the body $\theta = 80°c$ when $t = 0$. min.

The temperature of the body $\theta = 60°c$ when $t = 20$ min

The temperature of the air is 40°c.

By Newton's Law of cooling,

we have $\frac{d\theta}{dt} = -k(\theta - \theta_0)$.

where θ_0 is the temperature of the air.

$$\frac{d\theta}{dt} = -k(\theta - 40) \quad \text{---} (1)$$

Separate the variables and Integrate, we get

$$\int \frac{d\theta}{\theta - 40} = -k\int dt + \log c.$$

$$\log|\theta - 40| = -kt + \log c.$$

$$\log|\theta - 40| - \log c = -kt$$

$$\log\left|\frac{\theta - 40}{c}\right| = -kt$$

$$\frac{\theta - 40}{c} = e^{-kt}$$

$$\theta - 40 = c e^{-kt}$$

$$\theta = 40 + c e^{-kt} \quad \text{---} (2)$$

When $t = 0$, $\theta = 80°c$,

From (2), $80 = 40 + c e^{-k \cdot 0}$

$$c = 40.$$

Sub $c = 40$ in (2), we get $\theta = 40 + 40 e^{-kt}$ — (3)

When $t = 20$, $\theta = 60$

$$60 = 40 + 40 e^{-k20}$$

$$20 = 40 e^{-k20}$$

$$\frac{1}{2} = e^{-20k} \quad \text{——} (4).$$

When $t = 40$ min, $\theta =$ ______

From (3), we have, $\theta = 40 + 40 e^{-40k}$ —— (3)

From (4), $\frac{1}{2} = e^{-20k}$

$$\frac{1}{4} = e^{-40k} \quad \text{——} (5)$$

From (3) and (5), we get

$$\theta = 40 + 40 \cdot \frac{1}{4}$$

$$\theta = 50°c.$$

(2) If the temparature of the air is 20°c and the temparature of the body drops from 100°c to 80°c. In 10 min. What will be its temparature after 20 min. when will be the temparature 40°c.

Sol: Let θ be the temparature of the body at time t.

The temparature of the body $\theta = 100°$ when $t = 0$ min

The temparature of the body $\theta = 80°c$ when $t = 10$ min

The temparature of the air $\theta_0 = 20°c$.

By Newton's Law of cooling, we have.

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

where θ_0 is the temparature of the body.

$$\frac{d\theta}{dt} = -k(\theta - 20).$$

Separate the variables and integrate, we get.

$$\int \frac{d\theta}{\theta - 20} = -k\int dt + \log c.$$

$$\log|\theta - 20| = -kt + \log c.$$

$$\log|\theta - 20| - \log c = -kt.$$

$$\log\left|\frac{\theta - 20}{c}\right| = -kt.$$

$$\frac{\theta - 20}{c} = e^{-kt}$$

$$\theta = 20 + c\,e^{-kt}. \quad \text{——} (1)$$

At $t=0$, $\theta = 100$.

From (1), $100 = 20 + c\,e^{-k.0}$

$$c = 80.$$

$$(1) \Rightarrow \quad \theta = 20 + 80\,e^{-kt} \quad \text{——} (2).$$

When $t = 10$ min, $\theta = 80°c$,

From (2), $80 = 20 + 80\,e^{-10k}.$

$$60 = 80\,e^{-10k}.$$

$$\frac{3}{4} = e^{-10k} \quad \text{——} (3).$$

(i) When $t = 20$ min, $\theta =$ ______

From (2), $\theta = 20 + 80\,e^{-k.20}. \quad \text{——} (4)$

$$\theta = 20 + 80.\,e^{-20k}.$$

$$(3) \Rightarrow \quad \frac{3}{4} = e^{-10k}.$$

$$\frac{9}{16} = e^{-20k}. \quad \text{——} (5).$$

From (4) and (5), we get

$$\theta = 20 + 80.\frac{9}{16} = 65°c.$$

$\therefore$ The temparature of the body will be $65°c$ after 20 min.

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(ii) When $\theta = 40°c$, $t = $ ——

From (2), $\theta - 20 = 80 e^{-kt}$

$$40 - 20 = 80 e^{-kt}$$

$$20 = 80 e^{-kt}$$

$$\frac{1}{4} = e^{-kt}$$

$$\frac{1}{4} = \frac{1}{e^{kt}}$$

$$e^{kt} = 4.$$

$$[e^{k}]^{t} = 4 \quad \text{—— (6)}$$

From (5), $\frac{3}{4} = e^{-10k}$.

$$\frac{3}{4} = \frac{1}{e^{10k}}.$$

$$\frac{4}{3} = e^{10k}.$$

$$e^{k} = \left(\frac{4}{3}\right)^{1/10} \quad \text{—— (7)}$$

From (6) and (7), we get

$$\left[\left(\frac{4}{3}\right)^{1/10}\right]^{t} = 4.$$

Taking log. bothsides, we get

$$\log\left[\left(\frac{4}{3}\right)^{1/10}\right]^{t} = \log 4$$

$$t \cdot \frac{1}{10} \log \frac{4}{3} = \log 4$$

$$t = \frac{10 \log 4}{\log\left(\frac{4}{3}\right)} = $$

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→ A murder victim is discovered and a lieutenant from the Forensic science laboratory is summoned to estimate the time of death. The body is located in a room that is kept at a constant temparature of 68°F. The lieutenant arrived at 9.40 PM and measured the body temparature as 94.4°F at that time. Another measurement of the body temparature at 11 PM is 89.2°F Find the estimated time of death.

Sol:- Let θ be the temparature of the body at time t

Temparature of the room is $\theta_0 = 69°F$.

Temparature of the body at time $t=0$ (9.40 PM) is $\theta = 94.4$

Temparature of the body at time $t = 80$ min (11 PM) is $\theta = 89.2$.

Temparature of the human body is $98.6°F$.

Normal temparature of the human body is $98.6°F$.

We have to find the time $t =$ ___ when temparature $\theta = 98.6°F$

By Newton's Law of cooling, we have.

$$\frac{d\theta}{dt} \propto \theta - \theta_0$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

Separating the variables and integrate bothsides, we get

$$\int \frac{d\theta}{\theta - \theta_0} = -k\int dt + \log c$$

$$\log|\theta - \theta_0| - \log c = -kt$$

$$\log\left|\frac{\theta - \theta_0}{c}\right| = -kt$$

$$\frac{\theta - \theta_0}{c} = e^{-kt}$$

$$\theta - \theta_0 = c e^{-kt}$$

$$\theta = \theta_0 + c e^{-kt}$$

$$\theta = 68 + c e^{-kt} \quad \text{——— (1)}$$

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→ We have time $t=0$, temparature $\theta = 94.4$

From ①, $94.4 = 68 + ce^{-k(0)}$

$c = 94.4 - 68$

$c = 26.4$

① ⟹ $\theta = 68 + 26.4\, e^{-kt}$ —— ②.

→ We have time $t = 80$ min, temparature $\theta = 89.2$

From ②, $89.2 = 68 + 26.4\, e^{-80k}$

$26.4\, e^{-80k} = 89.2 - 68$

$e^{-80k} = \frac{21.2}{26.4}$

$e^{80k} = \frac{26.4}{21.2}$

$80k = \log\left(\frac{26.4}{21.2}\right)$

$k = 0.00274$

② ⟹ $\theta = 68 + 26.4\, e^{(-0.00274)t}$ —— ③

→ We have to find the time $t =$ —— when temparature $\theta = 98.6°F$

From ③, $98.6 = 68 + 26.4\, e^{(-0.00274)t}$

$26.4\, e^{(-0.00274)t} = 98.6 - 68$

$e^{(-0.00274)t} = \frac{30.6}{26.4}$

$(-0.00274)t = \log\left(\frac{30.6}{26.4}\right)$

$t = -53.88$

Death occured approximately 53.8 minutes before first measurement at 9.40.

This places the time of death approximately at 8.46 PM.

NEWTON'S LAW OF COOLING.

1 If the temparature of the air is 20°C and the temparature of the body drops from 100°C to 80°C in 10 minutes. What will be its tempa-rature after 20 minutes. When will be the temparature 40°C A:- 48.2 min

2 A murder victim is discovered and a lieutenant from the Forensic Science laboratory is summoned to estimate the time of death. The body is located in a room that is kept at a constant temparature 68°F. The lieutenant arrived at 9.40 P.M and measured the body temparature as 94.4°F at that time. Another measurement of the body temparature at 11 P.M is 89.2°F Find the estimated time of death.

Ans:- 8.46 P.M.

3 An object whose temparature is 75°C cools is an atmosphere of constant temparature 25°C at the rate $k\theta$, θ being the excess temparature of the body over the temparature. If after 10 minutes the tempara-ture of the object falls to 65°C, find its temparature after 20 min. Find the time required to cool down to 55°C Ans:- 23 min.

4 Water is heated to the boiling point temparature 100°C. It is then removed from heat and kept in a room which is at a constant temparature of 60°C. After 3 minutes, the temparature of the water is 90°C. Find the temparature after 6 min Ans:- 82.5°C.

5 A body of temparature 80°F is placed in a room of constant tempa-rature 50°F at a time t=0. At the end of 5 minutes the body has cooled to a temparature of 70°F. When will the temparature of the body be 60°F? Ans:- t = 13.55 min.

6 According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temparature of the substance and that of the air. If the temparature of the air is 40°C and the substance cools from 80°C to 60°C, 20 min. What will be the temparature of the substance after 40 minutes? Ans :- 49.86°C.

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7 A copper ball is heated to a temparature of 80°C. Then at time $t = 0$ it is placed in water which is maintained at 30°C. If at $t = 3$ min. the temparature of the ball is reduced to 50°C, find the time at which the temparature of the ball is 40°C Ans:- 5.27 min.

8 If the temparature of the air is 30°C and the substance cools from 100°C to 70°C in 15 min. find when the temparature will be 40°C Ans :- 52.5 min.

9 An object cools from 120°F to 95°F in half an hour when surrounded by air whose temparature is 70°F. Find its temparature at the end of another half an hour Ans :- 95.08°F.

10. The temparature of a cup of coffee is 92°C when freshly poured the room temparature being 24°C. In one minute it was cooled to 80°C How long a period must elapse, before the temparature of the cup becomes 65°C Ans:- 2.61 min.

11 Water at temparature 100°C cools in 10 min to 88°C in a room of temparature 25°C. Find the temparature of water after 20 min.
Ans :- 77.9°C

12 If the air is maintained at 30°C and the temparature of the body cools from 80°C to 60°C, 12 min. find the temparature of the body after (i) 36 min (ii) 24 min.

Law of Natural Growth or Decay :—

Let $x(t)$ be the amount of a substance at time t and let the substance be getting converted chemically. A law of chemical conversion states that the rate of change of amount $x(t)$ of a chemically changing substance is proportional to the amount of the substance available at that time.

i.e $\frac{dx}{dt} \propto x$ i.e. $\frac{dx}{dt} = -kx$.

where k is a constant of proportionality.

This differential equation can also describe in a simple way the population growth, radioactive decay etc.

If as t increases, x increases.

we can take. $\frac{dx}{dt} = kx \ (k>0)$

If as x decreases as t increases

we can take. $\frac{dx}{dt} = -kx \ (k>0)$

(1) In a chemical reaction a given substance is being converted into another at a rate proportional to the amount of substance unconverted. If $\left(\frac{1}{5}\right)^{th}$ of the original amount has been transformed in 4 minutes, how much time will be required to transform one half.

Sol:- Let x grams be the amount of the remaining substance after t minutes.

$\therefore$ The differential equation is $\frac{dx}{dt} = -kx$, $k>0$.

Separate the variables and integrate, we get

$$\int \frac{dx}{x} = -k\int dt + \log c$$

$$\log x - \log c = -kt$$

$$\log \frac{x}{c} = -kt$$

$$\frac{x}{c} = e^{-kt}$$

$$x = c e^{-kt} \quad \text{—— (1)}$$

Let the original amount of substance be m grams.

when $t=0$, $x = m$.

From (1), $m = c e^{-k\cdot 0}$

$$c = m$$

Sub. $c = m$ in (1), we get

$$x = m e^{-kt} \quad \text{—— (2)}$$

When $t = 4$, $x = m - \frac{m}{5} = \frac{4m}{5}$.

From (2), $x = m e^{-kt}$.

$$\frac{4m}{5} = m e^{-4k}$$

$$\frac{4}{5} = e^{-4k} \quad \text{—— (3)}.$$

We have to find t when $x = \frac{m}{2}$.

From (2), $\frac{m}{2} = m e^{-kt}$

$$\frac{1}{2} = e^{-kt}.$$

$$\frac{1}{2} = (e^{-k})^{t} \quad \text{—— (4)}$$

From (3), $\frac{4}{5} = (e^{-k})^4$.

$$e^{-k} = \left(\frac{4}{5}\right)^{1/4} \quad \text{—— (5)}$$

From (4) and (5), we get

$$\frac{1}{2} = \left(\frac{4}{5}\right)^{\frac{t}{4}}.$$

Taking log. bothsides, we get

$$\log\left(\frac{1}{2}\right) = \frac{t}{4}\log\left(\frac{4}{5}\right)$$

$$t = \frac{4\log\left(\frac{1}{2}\right)}{\log\left(\frac{4}{5}\right)}$$

$$t = 12.4 \text{ min}$$

(2) Bactesia in a culture grows exponentially so that the initial numbes has doubled in three hours. How many times the initial numbes will be present after 9 hours.

Sol:- Let Initially, at time $t=0$, the numbes of bactesia be A.

Let $N(t)$ be the numbes at time t. Since the bactesial grows exponentially.

The differential equation is $\frac{dN}{dt} = kN$

Separate the variables and integrate

$$\int \frac{dN}{N} = k\int dt + \log c$$

$$\log N = kt + \log c$$

$$\log \frac{N}{c} = kt$$

$$N = ce^{kt} \quad \text{—} (1)$$

At $t=0$, $N = A$.

From (1), $A = ce^{k \cdot 0}$

$c = A$.

$$\therefore N = Ae^{kt} \quad \text{—} (2)$$

At $t=3$, $N = 2A$.

From (2), $2A = Ae^{3k}$

$$2 = e^{3k} \quad \text{—} (3)$$

We have to find N at $t=9$.

From ②, $N = Ae^{kt}$

$$N = Ae^{9k}.$$

$$N = A(e^{k})^{9} \quad \text{— ④}$$

From ③, $2 = e^{3k}$

$$2 = (e^{k})^{3}$$

$$e^{k} = 2^{1/3} \quad \text{— ⑤}$$

From ④ and ⑤, we get

$$N = A \cdot (2^{1/3})^{9}$$

$$N = A\, 2^{3}$$

$$N = 8A.$$

$\therefore$ After 9 hours the bacteria will be 8 times that was present initially.

(3) A bacterial culture growing exponentially increases from 200 to 500 grams in the period from 6 a.m to 9 a.m. How many grams will be present at noon.

Sol:- Let N be the number of bacteria in a culture at any time $t > 0$,

The differential equation is $\frac{dN}{dt} = NK$.

Separate the variables and integrate, we get

$$\int \frac{dN}{N} = k \int dt + \log c$$

$$\log N - \log c = kt$$

$$\log \frac{N}{c} = kt$$

$$N = ce^{kt} \quad \text{— ①}$$

When $t=0$, $N = 200$ grams.

from ①, $N = ce^{kt}$.

$$200 = ce^{0.k}$$

$$c = 200$$

$$\therefore N = 200e^{kt} \text{ ——— ②}$$

When $t = 3$ hours, $N = 500$ grams

from ②, $N = 200\,e^{kt}$

$$\frac{500}{200} = e^{3k}$$

$$e^{3k} = \frac{5}{2}$$

$$3k \log_e e = \log \frac{5}{2}$$

$$k = \frac{1}{3}\log_e 2.5 = \log_e (2.5)^{1/3}$$

Hence the number of bacteria in the culture at any instant of time $t>0$ is given by.

$$N = 200\,e^{kt}$$

$$N = 200\,e^{\log(2.5)^{\frac{t}{3}}}$$

$$N = 200\,(2.5)^{t/3}.$$

$\therefore$ After 6 hours, the number of bacteria present will be

$$N = 200\,(2.5)^{6/3}$$

$$N = 200\,(2.5)^2$$

$$N = 200\,(6.25)$$

$$N = 1250 \text{ grams}.$$

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LAW OF NATURAL GROWTH OR DECAY

1 The mass of crystalline deposit increases at a rate which is proportional to its mass at that time. The deposit has stated around a crystal seed of 5 grams. Find an expression of its mass at time t. If in 30 minutes the mass of the deposit increases by 1 gram. what will be the mass of the deposit after 10 hours. Ans:- $5\left(\frac{6}{5}\right)^{20}$.

2 The rate at which a certain substance decomposes in a certain solution at any instant is proportional to the amount of it present in the solution at that instant. Initially, there are 27 grams and three hours later, it is found that 8 grams are left. How much substance will be left after one more hour. Ans:- $\frac{16}{3}$ grams.

3 The number x of bacteria in a culture grow at a rate proportional to x. The value of x was initially 50 and increased to 150 in 1 hour. what will be the value of x after $1\frac{1}{2}$ hour. Ans:- $50(3)^{1/2}$ grams.

4 The rate of growth of a bacteria is proportional to the number present. If initially there were 100 bacteria and the amount doubles in 1 hour. how many bacteria will be there after $2\frac{1}{2}$ hours. Ans:- 564.

5 In a certain reaction, the rate of conversion of a substance at time 't' is proportional to the quantity of the substance still untransformed at the instant. At the end of one hour 60 grams while at the end of 4 hours 21 grams remain. How many grams of the first substance was there initially? Ans:- 89 grams approx.

6 A radio active substance disintegrate at a rate proportional to its mass when mass is 10 mgm, the rate of disintegration is 0.051 mgm. per day. How long will it take for the mass to reduce from 10 to 5 mgms? Ans:- 135 days appro.

7 A bacterial culture growing, exponentially, increases from 100 to 400 grams in 10 hours. How many was present after 3 hours?

30

Ans:- 151.57.

8 If 30% of a radio active substance disappears in 10 days. how long will it take for 90% to disappear? Ans:- $10\left[\frac{\log 10}{\log 10-\log 7}\right]$

9 Under certain conditions can sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If 75 grams was there at time $t=0$ and 8 grams are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hr.

Ans:- 21.5 grams.

10 If 10% of 50 mg of a radio active material decays in 2 hours, find the mass of the material left at any time t and the time at which the material has decayed to one half of its initial mass. Ans:- 13 hrs.

11 If the population of a city gets doubled in 2 yrs and after 3 years the population is 15,000. find the initial population of the city

Ans:- 5297.

12 Bacteria in a certain culture increases at a rate proportional to the number present If the number N increases from 1000 to 2000 in one hour, how many are present at the end of 1.5 hour

Ans:- 2828 Appo.

13 In a culture yeast, the amount y of active yeast grows at a rate proportional to the amount present. If the original amount y doubles in 2 hours how long does it take for the original amount to triple Ans:- 3.17 hours

14 A bacterial culture population A is known to have a rate of growth proportional to A itself. Between noon and 2.P.M, If the population triples. at what time (no controls being exerted) should A become 100 times what it was at noon, given that. Ans:- 8.3837.

15 Find the half life of uranium which disintegrates at a rate proportional to the amount present at any instant. Given that m_1 and m_2 grams of uranium are present at time t_1 and t_2 respectively.

Ans:- $T = \dfrac{(t_2 - t_1)\log 2}{\log\left(\frac{m_1}{m_2}\right)}$

16 If radioactive carbon-14 has half life of 5750 years, what will remain of one gram after 3000 years? Ans:- 0.697 gm.

17 If was found that 0.5% of radium disappears in 12 years

(a) What percentage will disappear in 1000 years? Ans:- 34.2%.

(b) What is half life of radium? Ans:- 1672.18 years.

18 A radio active substance dis integrates at a rate proportional to the amount of the substance present. If 50% of the substance disintegrates in 1000 years approximately what percentage of the substance will dis integrate in 50 years Ans:- 3.5%.

19 A culture initially has N_0 number of bacteria. At $t = 1$ hour, the number of bacteria is measured to be $\frac{3}{2}N_0$. If the rate of growth is proportional to the number of bacteria present. determine the time necessary for the number of bacteria to triple Ans:- 2.71 hours.

20 Bacteria in a certain culture increases at a rate proportional to the number present. If the number doubles in one hour. how long does it take for the number to triple. Ans:- 1.58 hours.

21 In a chemical reaction a given substance is being converted into another at a rate proportional to the amount of substance unconverted. If $\left(\frac{1}{3}\right)^{th}$ of the original amount has been transformed in 4 min. how much time will be required to transform one half. Ans:- 13 min

22 A bacterial culture growing exponentially increases from 200 to 500 grams in the period from 6 a.m to 9 a.m. How many grams will be present at noon? Ans:- 1249.8 grams

23 Bacteria in a culture grows exponentially so that the initial number has doubled in 3 hrs. How many times the initial number will be present after 9 hrs.

Module - 2

Higher Order linear differential Equations :-

- Definition : An equation of the form

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1} y}{dx^{n-1}} + P_2(x)\frac{d^{n-1} y}{dx^{n-2}} + \cdots + P_n(x)y = Q(x)$$

Where $P_1(x), P_2(x) \ldots P_n(x)$ & $Q(x)$ are all real, Continuous functions of x defined on an Interval I is called linear differential equations of order n.

Eg:-

i) $x^2\frac{d^2y}{dx^2} + (x-2)\frac{dy}{dx} - 2y = x^3$ is a Second order linear differential Equation,

ii) $x^3y''' + 2x^2y'' + 2y = 10\left(x + \frac{1}{x}\right)$ is a third order D.E.

→ Linear differential equation with constant Coefficients :

An eq of the form

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1} y}{dx^{n-1}} + P_2\frac{d^{n-1} y}{dx^{n-2}} + P_n(y) = Q(x)$$

where $P_1, P_2 \ldots P_n$ are all real Constants. and $Q(x)$ is continuous function of x is called an ordinary linear D.E with Constant Coefficients.

Eg:- i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y - x^2$ is a 2^{nd} order linear D.E

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ii) $y''' + 2y'' + 3y' + 5y = 0$ is a 3^{rd} order linear D.E.

Note :-

1) In a linear D.E we can observe the following points

a) The dependent variable 'y' and its derivatives of any.

eg: $\frac{d^2y}{dx^2} + y\frac{dy}{dx} + 5y = x^3$.

⇒ derivatives in any term are not multiplied together.

⇒ Coefficients of derivatives are either functions of independent variable or constant terms.

eg: i) $x^3y''' + 2x^2y'' + 2y = 10\left(x+\frac{1}{x}\right)$

ii) $\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 4y = x^2$.

2) If a D.E is not linear then it is called non-linear D.E but a D.E of degree three need not be linear.

Eg:1 $\frac{d^2y}{dx^2} + 2x^2\frac{dy}{dx} + 2xy^2 - \sin x$ is degree 1 but it is not linear because in the third term, coefficient of y is $2xy$ instead a function of 'x' (or) the 3^{rd} term degree of y is two.

Eg:2 $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = e^x$ is of degree 1.

but not linear because in 2^{nd} term $\frac{dy}{dx}$ occur in 2^{nd} degree.

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Differential Operator - Notation :-

Let the differential operator $\frac{d}{dx}$ denoted by D and diff operators $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \cdots \frac{d^n}{dx^n}$ be denoted respectively by $D^2, D^3, \cdots D^n$. When applied on a function y of x yield. Thus $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2} \cdots, D^ny = \frac{d^ny}{dx^n}$

Let the n^{th} Order L.D.E be

$$\frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n y = Q(x)$$

The operator form of the above D.E is

$$D^ny + P_1 D^{n-1}y + P_2 D^{n-2}y + \cdots + P_n y = Q(x)$$

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n)y = Q(x)$$

which is of the form $f(D)y = Q(x)$;

where $\boxed{f(D) = D^n + P_1 D^{n-1} + \cdots P_n}$

Note :-

(i) If $Q(x) \neq 0$ for some x in I (any interval) then the equation $f(D)y = Q(x)$ is called a linear and non homogeneous D.E.

Eg:- $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x^2$ i.e $(D^2+4D+4)y = x^2$

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ii) If $Q(x)=0$ for some x in I then the eq $f(D)y=0$ is called a linear homogeneous equation.

Ex: $\frac{d^3y}{dx^3}+5\frac{d^2y}{dx^2}+4\frac{dy}{dx}-y=0$

i.e $(D^3+5D^2+4D)y=0$.

→ General solution of $f(D)y=0$

Let $f(D)y=0$ be the n^{th} order L.D.E. Let $y=y_1(x)$, $y=y_2(x)$, ---, $y=y_n(x)$ are n Linearly independent solutions of $f(D)y=0$, then.

$y=c_1y_1(x)$, $y=c_2y_2(x)$ --- $y=c_ny_n(x)$.

where $c_1, c_2, --- c_n$ are real constant are also. solutions of $f(D)y=0$ and then $y=c_1y_1+c_2y_2+--+c_ny_n$ is called the general solution of $f(D)y=0$.

Defination :-

If $y=y_p$ is a particular solution of $f(D)y=Q(x)$. Containing no arbitary constants and $y=y_c$ is the general solution of $f(D)y=0$. They $y=y_c+y_p$ is called the G.S of $f(D)y=Q$.

→ Complementary function Particular Integral of D.E of $f(D)y=0$

Let $y=y_c+y_p$ is the G.S of $f(D)y=Q(x)$ then the part

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y_c of the G.S is called the Complementary function of $f(D)y = Q$, And the part as y_p of the G.S is called the particular integral of $f(D)y = Q$.

Auxiliary Eq:

Consider the D.E $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots P_n)y = Q(x)$

which is of the form $f(D)y = Q(x)$

where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n$

The algebraic Eq $f(m) = 0$ i.e $m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0$ where $P_1, P_2, P_3 \cdots P_n$ are real Constants, is called the auxilary equation of $f(D)y = 0$

Since, the auxiliary eq $f(m) = 0$ is an algebraic eq of degree n, it will have n roots then, 3 Cases will arise.

Case-i: When the auxiliary eq has real & distinct roots.

Let $f(D)y = 0$ be the given L.D.E of order n.

The auxilary eq of $f(D)y = 0$ is $f(m) = 0$.

i.e $m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0$

Let $m_1, m_2, m_3, \cdots m_n$ be n real & distinct roots.

The G.S of $f(D)y = 0$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

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Ex 1. If 0, 1, -1 are the roots of an auxiliary eq $f(m)=0$, then the G.S of $f(D)y=0$ is $y=c_1e^{0x}+c_2e^{x}+c_3e^{-x}$ where c_1, c_2, c_3 are arbitrary constants.

∴ Order of the D.E $f(D)y=0$ is 3.

2. If 1, -1, 2, 3 are the roots of an auxiliary eq $f(m)=0$, then the G.S of $f(D)y=0$ is
$y=c_1e^{x}+c_2e^{-x}+c_3e^{2x}+c_4e^{3x}$
where c_1, c_2, c_3, c_4 are arbitrary constants,
Order of D.E $f(D)y=0$ is 4,
∵ The G.S have 4 arbitrary constants.

Q) Solve $\frac{d^2y}{dx^2}-y=0$.

Sol: G.T, $\frac{d^2y}{dx^2}-y=0$.

An operator form of the given D.E is $(D^2-1)y=0$
Which is of the form $f(D)y=0$
where $f(D)=D^2-1$.

An auxilary eq is $f(m)=0$. i.e $m^2-1=0$
$m=1,-1$.

The roots are real & distinct.
The G.S of D.E is $y=c_1e^{x}+c_2e^{-x}$.
where c_1 & c_2 are arbitrary constants.

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Q) Solve $(D^2-5D+6)y=0$

Sol: G.T. $(D^2-5D+6)y=0$

which is of the form $f(D)y=0$.

where $f(D)=(D^2-5D+6)$

An auxilary eq is $f(m)=0$ i.e. $m^2-5m+6=0$.

$m=2,3$.

The roots are real & distinct.

The G.S of D.E is $y=c_1e^{2x}+c_2e^{3x}$

where c_1 & c_2 are arbitary constants.

Case-2:-

When the auxilary eq has real & repeated roots.

Let $f(D)y=0$ be the given D.E of order 'n'.

The auxilary eq of $f(D)y=0$ is $f(m)=0$

i.e. $m^n+P_1m^{n-1}+P_2m^{n-2}+\cdots+P_n=0$.

Let $m_1, m_2, \ldots, m_n$ be 'n' real roots,

i) Let $f(m)=0$ have two equal roots $m_1=m_2$ & all other roots $m_3, m_4, \ldots, m_n$ are distinct.

Then the G.S of $f(D)y=0$ is

$$y=(c_1x^0+c_2x^1+c_3x^2)e^{m_1x}+c_3e^{m_3x}+c_4e^{m_4x}+\cdots c_ne^{m_nx}.$$

ii) Let $f(m)=0$ have 3 equal roots $m_1=m_2=m_3$ & all other distinct roots $m_4, m_5, \ldots m_n$

The G.S of $f(D)y=0$ is

$$y=(c_1x^0+c_2x+c_3x^2)e^{m_1x}+c_4e^{m_4x}+\cdots+c_ne^{m_nx}$$

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Ex:

1) If 1, 1, −2 are the roots of auxilary eq $f(m)=0$ then the G.S of $f(D)y=0$ is

$$y=(c_1x^0+c_2x^1)e^{x}+c_3e^{-2x}.$$

where c_1, c_2 & c_3 are arbitary Constants.

Order of the D.E is 3.

2) If 1, 1, 1, −3 are the roots of auxilary eq $f(m)=0$ then the G.S of $f(D)y=0$ is

$$y=(c_1x^0+c_2x^1+c_3x^2)e^{x}+c_4e^{-3x}$$

where c_1, c_2, c_3 & c_4 are auxilary constants.

Order of D.E is 4.

Since the G.S have 4 Constants.

Q) Solve $(D-1)^3y=0$.

Sol: G.T, $(D-1)^3y=0$

which is of the form. $f(D)y=0$

where $f(D)=(D-1)^3$

An auxilary eq is $f(m)=0$. i.e $(m-1)^3=0$,

$m=1,1,1$.

The roots are real & repeated.

The G.S of the D.E is $y=(c_1x^0+c_2x^1+c_3x^2)e^{x}$

where

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Q) Solve $(D^2-4)(D+1)^2 y = 0.$

Sol: G.T, $(D^2-4)(D+1)^2 y = 0.$

which is of the form $f(D)y=0.$

where $f(D) = (D^2-4)(D+1)^2$

An auxilary eq is $f(m)=0.$ i.e., $(m^2-4)(m+1)^2=0.$

$m^2-4=0 \quad (m+1)^2=0$

$m=-2,2 \quad m=-1,-1$

The roots are real & repeated.

The. G.S of the D.E is

$$y = (c_1x^0+c_2x^1)e^{-x} + c_3e^{-2x} + c_4e^{2x}.$$

where c_1, c_2, c_3 & c_4 are arbitary constants.

Q) Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0.$

Sol: G.T, $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

$(D^3-3D+2)y=0$

which is of the form $f(D)y=0.$

where $f(D) = D^3-3D+2$

An auxilary eq is $f(m)=0.$

$m^3-3m+2=0.$

$m=-2,1,1.$

The roots are real and repeated.

The G.S of diff eq is

$$y=(c_1x^0+c_2x^1)e^{x}+c_3e^{-2x}.$$

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Case - 3 :-

When the auxilary eq has complex roots. Let $f(D)y = 0$ with the n^{th} order linear D.E;

Let $f(m) = 0$, i.e, $m^n + p_1 m^{n-1} + P_2 m^{n-2} + \cdots P_n = 0$. be the auxilary eq.

Let $a+ib$, a, b are real & $b \neq 0$ be a Complex roots of $f(m) = 0$.

Since the coefficients of $f(m) = 0$ are real constants. The Complex roots occur in Conjugate pairs. Hence $a-ib$ is also a root of $f(m) = 0$.

Let the other real roots $f(m) = 0$ be $m_3, m_4 \cdots m_n$.

$\therefore$ The G.S of $f(D)y = 0$ is:

$$y = e^{ax}\left[c_1 \cos bx + c_2 \sin bx\right] + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}.$$

Ex: 1) Let $2+3i$, $2-3i$, -3 are the roots of an auxilary eq $f(m) = 0$ then the G.S of $f(D)y = 0$ is $y = e^{2x}\left[c_1 \cos 3x + c_2 \sin 3x\right] + c_3 e^{-3x}$.

2) Let $-4i$, $4i$, $-2, -2$ are the roots of an auxilary eq $f(m) = 0$ then the G.S of $f(D)y = 0$ is given by $y = e^{0x}\left[c_1 \cos 4x + c_2 \sin 4x\right] + (c_3 x^0 + c_4 x^1) e^{-2x}$.

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→ Let $m_1 = m_2 = a+ib$, $m_3 = m_4 = a-ib$ are the roots of auxilary eq then the roots of $f(D)y=0$ is

$$y = e^{ax}\left[(c_1x^0 + c_2x^1)\cos bx + (c_3x^0 + c_4x^1)\sin bx\right]$$

Ex: Let $m_1 = m_2 = 2+i$ and $m_3 = m_4 = 2-i$, are the roots of $f(m)=0$ then the G.S of $f(D)y=0$ is

$$y = e^{2x}\left[(c_1+c_2x)\cos x + (c_3+c_4x)\sin x\right].$$

Q) Solve $(D^3-1)y=0$.

Sol: G.T, $(D^3-1)y=0$.

which is of the form $f(D)y=0$.

where $f(D)y = D^3-1$

An auxilary eq is $f(m)=0$ i.e $m^3-1=0$

The roots are $m=1$, $m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

The roots are imaginary.

The G.S of $f(D)y=0$.

$$y = c_1e^{x} + e^{-\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right].$$

Note: $a^3-b^3 = (a-b)(a^2+b^2+ab)$

$(a^3+b^3) = (a+b)(a^2-ab+b^2)$

$a^2-b^2 = (a+b)(a-b)$

Q) Solve $(D^3+1)y=0$.

Sol: G.T, $(D^3+1)y=0$ which is of the form $f(D)y=0$.

Where $f(D) = D^3+1$

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An the auxiliary eq is $f(m)=0$
i.e $m^3+1=0$
$m=-1, \frac{1 \pm \sqrt{3}i}{2}$

The roots are imaginary.

The G.S is $y=e^{\frac{1}{2}x}\left[c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right]+c_3e^{-x}$.

Q) Solve $(D^4-1)(D+2)^2y=0$.

Sol: G.T, $(D^4-1)(D+2)^2y=0$. which is of the form

$f(D)y=0$.

$f(D)=(D^4-1)(D+2)^2$

an auxilary eq is $f(m)=0$.

i.e, $(m^4-1)(m+2)^2=0$.

$(m^2-1)(m^2+1)=0$

$m=1,-1,i,-i$ $\quad$ $m=-2,-2$.

The roots are imaginary.

The G.S of $f(D)y=0$. is

$y=c_1e^{x}+c_2e^{-x}+(c_3+c_4x)e^{-2x}+e^{0x}\left[c_5\cos x+c_6\sin x\right]$

Q) Solve $(D^6-1)y=0$.

Sol: G.T, $(D^6-1)y=0$ which is of the form

$f(D)y=0$

$f(D)=D^6-1$

An auxilary eq is $f(m)=0$ i.e

$m^6-1=0$

$$(m^3)^2 - 1 = 0$$
$$(m^3+1)(m^3-1) = 0.$$

$m^3+1=0$ $\qquad$ $m^3-1=0$

$m=-1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ $\qquad$ $m = 1,\ \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i.$

The roots are imaginary.

The G.S of $f(D)y=0$ is

$$y = c_1 e^{x} + c_2 e^{-x} + \left[c_3 \cos\frac{\sqrt{3}}{2} + c_4 \sin\frac{\sqrt{3}}{2}\right] e^{x/2} + e^{-x/2}\left(c_5 \cos\frac{\sqrt{3}}{2}x + c_6 \sin\frac{\sqrt{3}}{2}x\right)$$

Case-4 :

Let $f(D)y=0$ be the n^{th} order L.D.E.
Let $f(m)=0$ be an auxilary eq. Let $a+\sqrt{b}$ ((a,b) are real and $b>0$) be an irrational root of $f(m)=0$. Since irrational roots of $f(m)=0$ occur in conjugate pairs. Hence $a-\sqrt{b}$ is also root of $f(m)=0$.

Let $m_1 = a+\sqrt{b}$, $m_2 = a-\sqrt{b}$, $m_3, m_4, \ldots, m_n$ are real & distinct. then the G.S of $f(m)=0$ is

$$y = e^{ax}\left[c_1 \cosh\sqrt{b}\,x + c_2 \sinh\sqrt{b}\,x\right] + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}.$$

Q) Solve $(D^3 - 14D + 8)y = 0$

Sol: G.T, $(D^3+14D+8)y=0$ which is of the form $f(D)y=0$.

An auxilary eq of $f(D)y=0$ is $f(m)=0$.

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$m^3 - 14m + 8 = 0$

$m = -4\ ,\ 3.414\ ,\ 0.585.$

The roots are $m = -4\ ,\ m = 2 \pm \sqrt{2}$ (1)

The G.S is $f(D)y = 0$ is

$y = c_1 e^{-4x} + e^{2x}\left[c_2 \cosh\sqrt{2}x + c_3 \sinh\sqrt{2}x\right]$

$y = c_1 e^{-4x} + e^{2x}\left[c_2 \cosh\sqrt{2}x + c_3 \sinh\sqrt{2}x\right]$

Inverse Operator :

The operator D is called differential operator. The operator D^{-1} is called inverse of the D.O.(D). i.e if Q is any function of x defined on an interval I then $D^{-1}.Q$ or $\frac{1}{D}Q$ is called the integral of Q.

$$\frac{1}{D}Q = \int Q\,dx.$$

Note: D is the differential operator $\Rightarrow \frac{1}{D}$ is integral operator.

Ex: i) $\frac{1}{D^2}(e^{3x}) = \frac{1}{D}\left[\frac{1}{D} e^{3x}\right] = \frac{1}{D}\left[\int e^{3x} dx\right] = \frac{1}{D}\left[\frac{e^{3x}}{3}\right]$

$= \frac{1}{3}\int e^{3x} dx = \frac{1}{9}e^{3x}$.

ii) $\frac{1}{D^2}(\cos 3x) = \frac{1}{D}\left[\frac{1}{D}(\cos 3x)\right]$

$= \frac{1}{D}\left[\int (\cos 3x) dx\right] = \frac{1}{D}\left[\frac{\sin 3x}{3}\right] = \frac{1}{3}\int \sin 3x \, dx$

$= -\frac{\cos 3x}{9}$.

Theorem :-

If Q is a function of x defined on an interval I and α is a constant then the particular value

$$\frac{1}{D-\alpha} Q = e^{\alpha x}\int Q e^{-\alpha x} dx$$

Note: $\frac{1}{D+\alpha}(Q) = e^{-\alpha x}\int Q e^{+\alpha x} dx$

Ex: i) $\frac{1}{(D+1)(D-1)} x = \frac{1}{(D+1)}\left[\frac{1}{D-1} x\right]$

$\left[\text{W.K.T}, \frac{1}{D-\alpha} Q = e^{\alpha x}\int Q e^{-\alpha x} dx\right]$ $[\because \alpha = 1]$

$= \frac{1}{D+1}\left[e^{x}\int \underset{f}{x}\,\underset{g}{e^{-x}} dx\right]$

$= \frac{1}{D+1}\left[e^{x}\left[x\left(\frac{e^{-x}}{-1}\right) - 1\left(\frac{e^{-x}}{(-1)^2}\right)\right]\right]$

$= \frac{-1}{D+1}(x+1)$

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$$= -\left[e^{-x}\int \underset{f}{(x+1)}\, \underset{g}{e^{x}}\,dx\right]$$

$$= -e^{-x}\left[(x+1)(e^{x}) - (1)(e^{x})\right]$$

ii) $\frac{1}{D-1}\sin(e^{-x}) = \left[\frac{1}{(D-1)}\sin e^{-x}\right]$

$\left[\text{W.K.T } \frac{1}{D-\alpha}Q = e^{\alpha x}\int Q e^{-\alpha x}\,dx\right]$

$= \cancel{\frac{1}{D-1}}\left[\right. = e^{x}\int \underset{f}{(\sin e^{-x})}\,\underset{g}{e^{-x}}\,dx \qquad [\because \alpha = 1]$

$= \cancel{e^{x}\sin e^{-x}\left(\frac{e^{-x}}{-1}\right) - (\cos e^{-x})(e^{-x})}(e^{-x})$

put $e^{-x} = t$

$-e^{-x}dx = dt$

$e^{-x}dx = -dt.$

$$= -e^{x}\int \underset{f}{t}\,\underset{g}{\sin t}\,dt$$

$$= e^{-x}\left[t(-\cos t) - 1(-\sin t)\right]$$

$$= -e^{x}\left[-e^{-x}\cos e^{-x} + \sin e^{-x}\right]$$

iii) $\frac{1}{D+1}e^{e^{x}} = e^{-x}\int e^{e^{x}} e^{x}\,dx$

put $e^{x} = t$

$e^{x}dx = dt.$

$$= e^{-x}\int e^{t}\,dt$$

$$= e^{-x}e^{t}.$$

$$= e^{-x}e^{e^{x}}.$$

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Methods of finding particular integral in Some Special Cases :-

Type - (1)

Particular integral of $f(D)y = Q(x)$ when $Q(x) = e^{ax}$ where 'a' is a real constant.

case-i) when $f(a) \neq 0$.

Consider the D.E $f(D)y = Q$ where $Q = e^{ax}$

Particular integral $P.I = y_p = \frac{1}{f(D)} Q = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$

when $f(a) \neq 0$.

Ex: i) $\frac{1}{D^2-5D+6} e^x = \frac{e^x}{2}$

$f(D) = D^2-5D+6$, $Q = e^x$,

Here $a=1$

$f(a) = f(1) = 1^2-5(1)+6 = 2 \neq 0$.

ii) $\frac{1}{(D+2)^2(D+3)} e^{3x} = \frac{e^{3x}}{150}$

$f(D) = (D+2)^2(D+3)$, $Q = e^{3x}$.

Here $a=3$,

$f(a) = f(3) = (3+2)^2(3+3) = 150 \neq 0$.

Note: $\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$

$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

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iii) $\frac{1}{(D+2)^2}\cosh x = \frac{1}{(D+2)^2}\left(\frac{e^{x}+e^{-x}}{2}\right)$

$$= \frac{1}{2}\frac{1}{(D+2)^2}e^{x} + \frac{1}{2}\frac{1}{(D+2)^2}e^{-x}$$

Here $a=1$

$$= \frac{1}{2}\cdot\frac{1}{(1+2)^2}e^{x} + \frac{1}{2}\frac{1}{(-1+2)^2}e^{-x}$$

$$= \frac{e^{x}}{18} + \frac{e^{-x}}{2}.$$

Case ii)

When $f(a) = 0$.

$$P.I = y_p = \frac{1}{f(D)}e^{ax} = \frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)^k\phi(D)}e^{ax}$$

$$= \frac{x^k}{k!}\frac{e^{ax}}{\phi(a)} \quad \text{when } \phi(a)\neq 0$$

$$\left[\begin{array}{l}\text{When } f(a)=0,\ f(D)=(D-a)^k\phi(D)\\ (D-a)^k \text{ is factor of } f(D), \text{ and } \phi(a)\neq 0\end{array}\right]$$

i) $\frac{1}{D^2-5D+6}e^{2x} = \frac{1}{(D-2)(D-3)}e^{2x} = \frac{x^1}{1!}\frac{e^{2x}}{-1} = -xe^{2x}.$

$f(D) = D^2-5D+6$, $a=2$, $f(a)=f(2)=0$

$f(D) = (D-2)(D-3)$

$(D-2)$ is factor of $f(D)$, $k=1$

$\phi(D) = D-3$, $\phi(a) = \phi(2) = 2-3 = -1 \neq 0.$

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ii) $\frac{1}{(D-2)^4} e^{2x} = \frac{x^4}{4!} \cdot \frac{e^{2x}}{1} = \frac{x^4 e^{2x}}{24}$

iii) $\frac{1}{(D+2)^2(D-3)} e^{-2x} = \frac{x^2}{2!} \cdot \frac{e^{-2x}}{-5} = \frac{-x^2 e^{-2x}}{10}$.

$f(D) = (D+2)^2(D-3)$, $a = -2$

$f(-2) = 0$

$(D+2)^2$ is factor of $f(D)$., $k = 2$.

$\phi(D) = D-3$, $\phi(a) = \phi(-2) = -2-3 = -5 \neq 0$.

Q) Find the particular integral of $(D^2+6D+9)y = 2\sinh x$,

Sol: G.T , $(D^2+6D+9)y = 2\sinh x$

which is of the form $f(D)y = Q$.

Here $f(D) = D^2+6D+9$

$Q = 2\sinh x. = 2\left(\frac{e^x - e^{-x}}{2}\right)$

$Q = e^x - e^{-x}$

$P.I = y_p = \frac{1}{f(D)} Q$.

$y_p = \frac{1}{D^2+6D+9}(e^x - e^{-x})$

$y_p = \frac{1}{(D+3)^2} e^x - \frac{1}{(D+3)^2} e^{-x}$

$y_p = \frac{1}{(1+3)^2} e^x - \frac{1}{(-1+3)^2} e^{-x}$

$y_p = \frac{e^x}{16} - \frac{e^{-x}}{4}$

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Q) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2\cosh x$, $y(0)=0$, $y'(0)=1$

Sol: G.T, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2\cosh x$, $y(0)=0$, $y'(0)=1$

An operator form of the given D.E is

$$(D^2+4D+5)y = 2\cosh x$$

Here $f(D) = D^2+4D+5$

$$Q = 2\cosh x.$$

$$Q = 2\left(\frac{e^x+e^{-x}}{2}\right) = e^x+e^{-x}.$$

An auxilary eq is $f(m)=0$ i.e. $m^2+4m+5=0$.

$$m = \frac{-4\pm\sqrt{16-4(5)}}{2.1}$$

$$m = \frac{-4\pm 2i}{2}$$

$$m = -2\pm i$$

$\therefore$ The roots are imaginary.

$$C.F = y_c = e^{-2x}(C_1\cos x + C_2\sin x)$$

$$P.I = y_p = \frac{1}{f(D)}Q$$

$$y_p = \frac{1}{D^2+4D+5}(e^x+e^{-x})$$

$$y_p = \frac{1}{D^2+4D+5}e^x + \frac{1}{D^2+4D+5}e^{-x}.$$

$$y_p = \frac{1}{1^2+4(1)+5}e^x + \frac{1}{(-1)^2-4+5}e^{-x}$$

$$y_p = \frac{e^x}{10} + \frac{e^{-x}}{2}$$

The G.S is $y = y_c + y_p$.

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$$y = e^{-2x}(c_1\cos x + c_2\sin x) + \frac{e^x}{10} + \frac{e^{-x}}{2} \quad \text{—①.}$$

we have $\underset{x}{y}(0) = \underset{y}{0}$ i.e. $y=0$ when $x=0$.

From ①, $0 = c_1 + \frac{1}{10} + \frac{1}{2}$

$$c_1 = -\frac{(6)}{10} = \frac{-3}{5}$$

we have $y'(0) = 1$

i.e $y' = 1$ when $x=0$

diff w.r.t 'x', we get

$$y' = e^{-2x}(-c_1\sin x + c_2\cos x) - 2e^{-2x}[c_1\cos x + c_2\sin x] + \frac{e^x}{10} - \frac{e^{-x}}{2} \quad \text{—②}$$

From ②,

$$1 = c_2 - 2c_1 + \frac{1}{10} - \frac{1}{2}$$

$$1 = c_2 + \frac{6}{5} - \frac{4}{10}$$

$$1 = c_2 + \frac{8}{10}$$

$$1 - \frac{8}{10} = c_2$$

$$\frac{2}{10} = c_2$$

$$c_2 = \frac{1}{5}$$

Sub the values of c_1 & c_2 in ①, we get

$$y = e^{-2x}\left(\frac{-3}{5}\cos x + \frac{1}{5}\sin x\right) + \frac{e^x}{10} + \frac{e^{-x}}{2}$$

which is the Particular Solution of given D.E.

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$\underline{\text{Note}}$:

$\cancel{a = e^{\log_e}}$

$a = e^{\log_e a}$

$a^x = \left(e^{\log_e a}\right)^x = e^{(\log_e a)x}$

Q) Solve $(D^2 - 2D + 1)y = (1 + e^{-x})^2 + 2^{-x}$

$\underline{\text{Sol}}$: G.T, $(D^2 - 2D + 1)y = (1 + e^{-x})^2 + 2^{-x}$.

which is of the form $f(D)y = Q$

Here $f(D) = (D-1)^2$, $Q = (1 + e^{-x})^2 + 2^{-x}$

$$Q = 1 + e^{-2x} + 2e^{-x} + 2^{-x}$$

$$Q = 1 + e^{-2x} + 2e^{-x} + e^{-(\log_e 2)x}$$

$$Q = e^{0x} + e^{-2x} + 2e^{-x} + e^{-(\log_e 2)x}$$

The auxilary eq is $f(m) = 0$.

i.e. $(m-1)^2 = 0$

$m = 1, 1$

The roots are real & repeat.

$CF = y_c = (c_1 x^0 + c_2 x^1)e^x$

$P.I = y_p = \frac{1}{f(D)} Q$

$$y_p = \frac{1}{(D-1)^2}\left[e^{0x} + e^{-2x} + 2e^{-x} + e^{-(\log_e 2)x}\right]$$

$$= \frac{1}{(D-1)^2} e^{0x} + \frac{1}{(D-1)^2} e^{-2x} + 2\frac{1}{(D-1)^2} e^{-x} + \frac{1}{(D-1)^2} e^{-(\log_e 2)x}$$

$$= \frac{1}{(0-1)^2} e^{0x} + \frac{1}{(-2-1)^2} e^{-2x} + 2\frac{1}{(-1-1)^2} e^{-x} + \frac{1}{(-\log_e 2 - 1)^2} e^{-(\log_e 2)x}$$

$$\ldots e^{-2x} + \frac{e^{-x}}{2} + \frac{1}{(1 + \log_e 2)^2} e^{-(\log_e 2)x}$$

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$\therefore$ The G.S is $y = y_c + y_p$

$$y = (c_1 + c_2 x)e^{x} + 1 + \frac{e^{-2x}}{9} + \frac{e^{-x}}{2} + \frac{1}{(1+\log_e^2)^2} e^{-(\log_e 2)x}$$

Q) Solve $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$

Sol: G.T, $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$

which is of the form $f(D)y = Q$

Here, $f(D) = D^3 - 5D^2 + 7D - 3$

$Q = e^{2x}\cosh x$

An auxilary eq is $f(m) = 0$.

$m^3 - 5m^2 + 7m - 3 = 0$

$(m-1)(m^2 - 4m + 3) = 0$

$(m-1)(m-1)(m-3) = 0$.

$m = 1, 1, 3$

The roots are real & repeat.

$C.F = y_c = (c_1 x^0 + c_2 x^1)e^{x} + c_3 e^{3x}$

$P.I = y_p = \frac{1}{f(D)} Q$

$$y_p = \frac{1}{D^3 - 5D^2 + 7D - 3}\left[e^{2x}\cosh x\right]$$

$$= \frac{1}{D^3 - 5D^2 + 7D - 3} e^{2x}\left(\frac{e^{x} + e^{-x}}{2}\right)$$

$$= \frac{1}{D^3 - 5D^2 + 7D - 3}\frac{e^{3x}}{2} + \frac{1}{D^3 - 5D^2 + 7D - 3}\frac{e^{x}}{2}$$

$$= \frac{x}{2}\frac{1}{3D^2 - 10D + 7} e^{3x} + \frac{x}{2}\frac{1}{3D^2 - 10D + 7} e^{x}$$

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$$= \frac{x}{8}e^{3x} + \frac{x^2}{2}\frac{1}{6D-10}e^{x}$$

$$= \frac{x}{8}e^{3x} - \frac{x^2}{8}e^{x}$$

$\therefore$ The G.S is $y = y_c + y_p$

$$y = (c_1x^0 + c_2x^1)e^{x} + c_3e^{3x} + \frac{x}{8}e^{3x} - \frac{x^2}{8}e^{x}.$$

Q) Find the general solution of $(D-1)^4 y = e^x$.

Sol: G.T, $(D-1)^4 y = e^x$

which is of the form $f(D)y = Q$

$f(D) = (D-1)^4$

$Q = e^x$

An auxilary eq is $f(m) = 0$.

$$(m-1)^4 = 0$$

$$m = 1, 1, 1, 1$$

The roots are real & repeat.

$$C.F = y_c = (c_1x^0 + c_2x^1 + c_3x^2 + c_4x^3)\,e^{x}$$

$$P.I = y_p = \frac{1}{f(D)}Q$$

$$y_p = \frac{1}{(D-1)^4}e^{x}$$

$$y_p = \frac{x^4}{4!}e^{x}.$$

$$y_p = \frac{x^4}{24}e^{x}.$$

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The G.S is $y = y_c + y_p$

$$y = (c_1x^0 + c_2x^1 + c_3x^2 + c_4x^3)e^x + \frac{x^4}{24}e^x.$$

Q) Solve $(D^2+4D+5)y = -2\cosh x + 2^x$.

Sol: G.T, $(D^2+4D+5)y = -2\cosh x + 2^x$

which is of the form $f(D)y = Q$

$f(D) = D^2+4D+5$

$\left(a^x = e^{x\log_e a}\right)$

$Q = -2\cosh x + 2^x$

$Q = -2\left(\frac{e^x + e^{-x}}{2}\right) + e^{(\log_e 2)x}$

$Q = -(e^x + e^{-x}) + e^{(\log_e 2)x}$.

An auxiliary eq is $f(m) = 0$

$m^2+4m+5 = 0$

$m = -2 \pm i$.

The roots are imaginary.

$C.F = y_c = e^{-2x}[c_1\cos x + c_2\sin x]$

$= e^{-2x}(c_1\cos x + c_2\sin x)$

$P.I = y_p = \frac{1}{f(D)} Q$.

$y_p = \frac{1}{D^2+4D+5}\left[-(e^x+e^{-x}) + e^{(\log_e 2)x}\right]$

$y_p = \frac{1}{D^2+4D+5}(-e^x) - \frac{1}{D^2+4D+5}e^{-x} + \frac{1}{D^2+4D+5}e^{(\log_e 2)x}$ —— (1)

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$$\frac{-e^{x}}{D^2+4D+5} = \frac{\cancel{x}}{\cancel{2D+4}} e^{x} = \frac{-e^{x}}{1+4+5} = \frac{-e^{x}}{10.} \quad \text{—(2)}$$

Here $a=1$

$f(a) = 1^2+4+5 = 10.$

$$\frac{e^{-x}}{D^2+4D+5} = \frac{e^{-x}}{2} \quad \text{—(3)}$$

$a=-1$

$f(a) = (-1)^2-4+5 = 1-4+5 = 2. \quad \cancel{(3)}$

$$\frac{1}{D^2+4D+5} e^{(\log_e 2)x} = \frac{1}{(\log_e 2)^2+4\log_e 2+5} e^{(\log_e 2)x}. \quad \text{—(4)}$$

Sub eq (2), (3) & (4) in (1).

$$y_p = \frac{-e^{x}}{10} - \frac{e^{-x}}{2} + \frac{1}{(\log_e 2)^2+4\log_e 2+5} e^{(\log_e 2)x}$$

The G.S is given by,

$y = y_c + y_p$

$$y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{e^{x}}{10} - \frac{e^{-x}}{2} + \frac{1}{(\log_e 2)^2+4\log_e 2+5} e^{(\log_e 2)x}$$

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Type - 2 :

To find the particular integral of $f(D)y = Q$ where $Q = \sin bx$ or $\cos bx$,

Consider the D.E of the form $f(D)y = Q$ where $Q = \sin bx$ or $\cos bx$.

Case-i) Let $f(D) = \phi(D^2) = D^2 + b^2$, and $\phi(-b^2) = 0$.

a) $y_p = \frac{1}{f(D)} Q = \frac{1}{D^2+b^2} \sin bx = \frac{-x}{2b} \cos bx$

$-b^2 + b^2 = 0.$

$= 0$

Ex: $\frac{1}{D^2+9} \sin 3x = \frac{-x}{2 \cdot 3} \cos 3x = \frac{-x \cos 3x}{6}$

$-3^2 + 9$

$\frac{1}{D^2+1} \sin x = \frac{-x}{2 \cdot 1} \cos x = \frac{-x \cos x}{2}$

$-1^2 + 1$

b) $y_p = \frac{1}{f(D)} Q = \frac{1}{D^2+b^2} \cos bx = \frac{x}{2b} \sin bx$

Ex: i) $\frac{1}{D^2+16} \cos 4x = \frac{x}{2 \cdot 4} \sin 4x = \frac{x \sin 4x}{8}$

$-4^2 + 16$

ii) $\frac{1}{D^2+1} \cos x = \frac{x}{2 \cdot 1} \sin x = \frac{x \sin x}{2}$

$-1^2 + 1 = 0$

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Case-ii) Let $f(D) = \phi(D^2) = D^2+b^2$ and $\phi(-b^2) \neq 0$.

a) $y_p = \frac{1}{f(D)}Q = \frac{1}{D^2+b^2}\sin bx = \frac{1}{\phi(-b^2)}\sin bx$

Ex: i) $\frac{1}{D^2+9}\sin 2x = \frac{1}{-2^2+9}\sin 2x = \frac{\sin 2x}{5}$

ii) $\frac{1}{D^2+4}\sin 3x = \frac{1}{-3^2+4}\sin 3x = \frac{-\sin 3x}{5}$

b) $y_p = \frac{1}{f(D)}Q = \frac{1}{D^2+b^2}\cos bx = \frac{\cos bx}{\phi(-b^2)}$

Ex: i) $\frac{1}{D^2+1}\cos 2x = \frac{\cos 2x}{-2^2+1} = \frac{\cos 2x}{-3}$

$\frac{1}{D^2+5}\cos 4x = \frac{1}{-4+5}\cos 4x = \frac{-\cos 4x}{11}$

Case-iii)

When $f(D)$ involving odd powers of D, for finding particular integral first we replace D^2 with '$-b^2$' and then rationalize to find the particular integral

Ex: i) $\frac{1}{D^2+D+1}\cos 2x = \frac{1}{-2^2+D+1}\cos 2x$

$= \frac{1}{D-3}\cos 2x$

$= \frac{(D+3)}{(D-3)(D+3)}\cos 2x$

$= \frac{D+3}{D^2-9}\cos 2x = \frac{(D+3)}{-2^2-9}\cos 2x$

$= \frac{-1}{13}\left[D(\cos 2x) + 3\cos 2x\right]$

$= \frac{-1}{13}\left[\frac{d}{dx}(\cos 2x) + 3\cos 2x\right]$

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$$= \frac{-1}{13}\left[-2\sin 2x + 3\cos 2x\right]$$

Note :

$\sin(A+B) = \sin A\cos B + \cos A\sin B$ ✓

$\sin(A-B) = \sin A\cos B - \cos A\sin B$ ✓

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$ ✓

$\cos(A-B) = \cos A\cos B + \sin A\sin B$ ✓

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\cos 3x = 4\cos^3 x - 3\cos x$

$\cos^3 x = \frac{3\cos x + \cos 3x}{4}$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

Note: When $Q(x) = \sin^2 x$ (or) $\cos^2 x$ we write $Q(x)$ interms of $\cos 2x$.

When $Q(x) = \sin^3 x$ or $\cos^3 x$; we write $Q(x)$ interms of $\sin 3x$ or $\cos 3x$

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when $Q(x) = \sin ax \cos bx$ or $\cos ax \cos bx$ or $\sin ax \sin bx$ then we write $Q(x)$ addition or subtraction of Sin and Cosine terms.

Q) Find the particular integral of $(D^4+4D^2+4)y = 2\cos^2 x$

Sol: G.T, $(D^4+4D^2+4)y = 2\cos^2 x$

which is of the form $f(D)y = Q$

where $f(D) = D^4+4D^2+4$

$$Q = 2\cos^2 x.$$

$$Q = 2\left(\frac{1+\cos 2x}{2}\right)$$

$$Q = e^{0.x} + \cos 2x$$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$y_p = \frac{1}{D^4+4D^2+4}\left(e^{0x} + \cos 2x\right)$$

$b = 2$

$D^2 = -b^2$

$$y_p = \frac{1}{D^4+4D^2+4} e^{0x} + \frac{1}{D^4+4D^2+4} \cos 2x$$

$$= \frac{1}{0^4+4(0)^2+4} e^{0x} + \frac{1}{(-2^2)^2+4(-2)^2+4} \cos 2x$$

$$= \frac{1}{4} + \frac{\cos 2x}{4}$$

Q) Solve $(D^2+9)y = e^{3x} + \cos^3 x$

Sol: G.T, $(D^2+9)y = e^{3x} + \cos^3 x.$

which is of the form $f(D)y = Q$

where $f(D) = D^2+9$

$$Q = e^{3x} + \cos^3 x = e^{3x} + \frac{3\cos x + \cos 3x}{4}$$

$$Q = e^{3x} + \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$

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An auxilary eq is $f(m)=0$

$$m^2+9=0$$

$$m=\pm 3i$$

The roots are imaginary

$$C.F = y_c = e^{0x}[c_1\cos 3x + c_2\sin 3x]$$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$= \frac{1}{D^2+9}\left(e^{3x} + \frac{3}{4}\cos x + \frac{1}{4}\cos 3x\right)$$

$$= \frac{1}{D^2+9}e^{x} + \frac{3}{4}\cdot\frac{1}{D^2+9}\cos x + \frac{1}{4}\cdot\frac{1}{D^2+9}\cos 3x$$

$$= \frac{1}{18}e^{3x} + \frac{3}{4}\cdot\frac{1}{8}\cos x + \frac{1}{4}\cdot\frac{x}{6}\sin 3x$$

The G.S is $y = y_c + y_p$

$$y = e^{0x}[c_1\cos 3x + c_2\sin 3x] + \frac{1}{18}e^{3x} + \frac{3}{32}\cos x + \frac{1}{4}\cdot\frac{x}{6}\sin 3x$$

Q) Solve $(D^2+5D-6)y = 2\sin 4x\cdot\sin x + e^{-x} + 2^x$

Sol: G.T, $(D^2+5D-6)y = 2\sin 4x\cdot\sin x + e^{-x} + 2^x$.

which is of the form $f(D)y = Q$.

Here $f(D) = D^2+5D-6$

$$Q = 2\sin 4x\cdot\sin x + e^{-x} + 2^x.$$

$$Q = \cos 3x - \cos 5x + e^{-x} + e^{(\log_e 2)x}$$

An auxilary eq is $f(m)=0$

i.e, $m^2+5m-6=0$.

$$m=-6, +1$$

The roots are real & distinct.

$$C.F = y_p = \frac{1}{f(D)} Q.$$

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$$y_p = \frac{1}{D^2+5D-6}\left[\cos 3x - \cos 5x + e^{-x} + e^{(\log_e 2)x}\right]$$

$$= \frac{1}{D^2+5D-6}\cos 3x - \frac{1}{D^2+5D-6}\cos 5x + \frac{1}{D^2+5D-6}e^{-x} + \frac{1}{D^2+5D-6}e^{(\log_e 2)x} \quad \text{--- (1)}$$

$$\frac{1}{D^2+5D-6}e^{-x} = \frac{1}{(-1)^2+5(-1)-6}e^{-x} = \frac{-e^{-x}}{10} \quad \text{--- (2)}$$

$$\frac{1}{D^2+5D-6}e^{(\log_e 2)x} = \frac{1}{(\log_e 2)^2+5(\log_e 2)-6}e^{(\log_e 2)x} \quad \text{--- (3)}$$

$$\frac{1}{D^2+5D-6}\cos 3x = \frac{1}{-3^2+5D-6}\cos 3x$$

$$= \frac{1}{5D-15}\cos 3x = \frac{1}{5}\cdot\frac{1}{D-3}\cos 3x$$

$$= \frac{1}{5}\cdot\frac{D+3}{(D-3)(D+3)}\cos 3x$$

$$= \frac{1}{5}\cdot\frac{D+3}{D^2-9}\cos 3x$$

$$= \frac{1}{5}\,\frac{D+3}{-3^2-9}\cos 3x$$

$$= \frac{-1}{90}\left[D(\cos 3x + 3\cos 3x)\right]$$

$$= \frac{-1}{90}\left[-3\sin 3x + 3\cos 3x\right] \quad \text{--- (4).}$$

$$\frac{1}{D^2+5D-6}\cos 5x = \frac{1}{-5^2+5D-6}\cos 5x$$

$$= \frac{1}{5D-31}\cos 5x$$

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$$= \frac{(5D+31)}{(5D-31)(5D+31)} \cos 5x$$

$$= \frac{5D+31}{25D^2-961} \cos 5x$$

$$= \frac{5D+31}{25(-5)^2-961} \cos 5x$$

$$= \frac{-1}{1586}\left[5D(\cos 5x)+31\cos 5x\right]$$

$$= \frac{-1}{1586}\left[-25\sin 5x+31\cos 5x\right] \quad \text{—} (5)$$

Sub (2), (3), (4), (5) in (1), we get

$$y_p = \frac{1}{30}\left[\sin 3x-\cos 3x\right]+\frac{1}{1586}\left[-25\sin 5x+31\cos 5x\right]$$
$$+ \frac{e^{-x}}{10} + \frac{e^{(\log_e 2)x}}{(\log_e 2)^2+5(\log_e 2)-6}$$

The G.S is $y = y_c + y_p$

$$y = c_1 e^{-6x} + c_2 e^{x} + \frac{1}{30}\left[\sin 3x-\cos 3x\right]$$
$$+ \frac{1}{1586}\left[-25\sin 5x+31\cos 5x\right] - \frac{e^{-x}}{10} + \frac{e^{(\log_e 2)x}}{(\log_e 2)^2+5(\log_e 2)-6}$$

Q) Solve $(D^2+4)y = (1+\cos x)^2 + e^{-x}$.

Sol: G.T, $(D^2+4)y = (1+\cos x)^2 + e^{-x}$.

which is of the form $f(D)y = Q$

Here $f(D) = D^2+4$

$$Q = (1+\cos x)^2 + e^{-x}$$

$$Q = 1+\cos^2 x + 2\cos x + e^{-x}$$

~~$Q = \sin^2 x + 2\cos x + e^{-x}$~~

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$$Q = e^{-x} + e^{0.x} + 2\cos x + \frac{1+\cos 2x}{2}$$

$$Q = e^{-x} + e^{0x} + 2\cos x + \frac{e^{0.x}}{2} + \frac{1}{2}\cos 2x$$

$$Q = e^{-x} + \frac{3}{2}e^{0x} + 2\cos x + \frac{1}{2}\cos 2x.$$

The auxilary eq is $f(m)=0$ i.e $m^2+4=0$

$m = \pm 2i$

The roots are imaginary,

$$C.F = y_c = \cancel{\frac{1}{f(D)}Q} = \cancel{\frac{1}{D^2+4}}$$

$$e^{0x}[c_1\cos 2x + c_2\sin 2x]$$

$$P.I = y_p = \frac{1}{f(D)}Q.$$

$$y_p = \frac{1}{D^2+4}\left(e^{-x} + \frac{3}{2}e^{0x} + 2\cos x + \frac{1}{2}\cos 2x\right)$$

$$y_p = \frac{1}{D^2+4}e^{-x} + \frac{3}{2}\cdot\frac{1}{D^2+4}\cdot e^{0.x} + 2\cdot\frac{1}{D^2+4}\cos x + \frac{1}{2}\cdot\frac{1}{D^2+4}\cos 2x$$

$$\frac{1}{D^2+4}e^{-x} = \frac{e^{-x}}{(-1)^2+4} = \frac{e^{-x}}{5} \quad \text{—②.}$$

$f(D) = D^2+4$, $a=-1$

$f(a) = f(-1) = (-1)^2+4 = 5 \neq 0.$

$$\frac{1}{D^2+4}e^{0x} = \frac{e^{0x}}{4} \quad \text{—③}$$

$a=0.$

$f(a) = f(0) = 4 \neq 0.$

$$\frac{1}{D^2+4}\cos x = \frac{1}{-1^2+4}\cos x = \frac{\cos x}{3} \quad \text{—④.}$$

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$$\frac{1}{D^2+4}\cos 2x = \frac{x}{2(2)}\sin 2x = \frac{x}{4}\sin 2x \quad \text{⑤}$$

Sub ②, ③, ④ & ⑤ in ①.

$$y_p = \frac{e^{-x}}{5} + \frac{3}{2}\frac{e^{0x}}{4} + \frac{2}{3}\cos x + \frac{x}{8}\sin 2x$$

The G.S is $y = y_c + y_p$.

$$y = e^{0x}\left[c_1\cos 2x + c_2\sin 2x\right] + \frac{e^{-x}}{5} + \frac{3}{2}\frac{e^{0x}}{4} + \frac{2}{3}\cos x + \frac{x}{8}\sin 2x$$

Q) Solve the D.E $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$

Sol: G.T, $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$

An operator form of given D.E is

$(D^3+4D)y = \sin 2x$ —①

which is in the form of $f(D)y = Q$

$f(D) = D^3+4D$

$Q = \sin 2x$

An auxilary eq is $f(m) = 0$

$m^3+4m = 0$

$m(m^2+4) = 0$

$m = 0, \pm 2i$

The roots are imaginary.

$$C.F = y_c = c_1e^{0x} + e^{0x}(c_2\cos 2x + c_3\sin 2x)$$

$$P.I = y_p = \frac{1}{f(D)}Q$$

$$y_p = \frac{1}{D^3+4D}\sin 2x$$

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$= \frac{1}{D^3 \cdot D + 4 \cdot D} \sin 2x$

$= x \cdot \frac{1}{3D^2 + 4} \sin 2x$

$= x \cdot \frac{1}{3(-2)^2 + 4} \sin 2x$

$= \frac{-x}{8} \sin 2x$

The G.S is $y = y_c + y_p$

$y = c_1 e^{0x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x) - \frac{x}{8} \sin 2x.$

Q) Solve $(D^2+9)y = \cos^3 x,$

Sol: G.T, $(D^2+9)y = \cos^3 x$

which is of the form $f(D)y = Q$

$f(D) = D^2+9$

$Q = \cos^3 x = \frac{3\cos x + \cos 3x}{4}$

The auxilary eq. is $f(m) = 0.$

$m^2 + 9 = 0.$

$m = \pm 3i$

$C.F = y_c = e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$

$P.I = y_p = \frac{1}{f(D)} Q$

$y_p = \frac{1}{D^2+9} \frac{3\cos x + \cos 3x}{4}$

$y_p = \frac{3}{4} \frac{1}{D^2+9} \cos x + \frac{1}{4} \frac{1}{D^2+9} \cos 3x$ —— (1)

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$$\frac{1}{D^2+9}\cos x = \frac{1}{-1+9}\cos x = \frac{1}{8}\cos x \quad \text{--- (2)}$$

$$\frac{1}{D^2+9}\cos 3x = \frac{x}{2.3}\sin 3x = \frac{x}{6}\sin 3x \quad \text{--- (3)}$$

sub eq (2) & (3) in (1).

$$y_p = \frac{3}{4}\cdot\frac{1}{8}\cos x + \frac{1}{4}\cdot\frac{x}{6}\sin 3x$$

$$y_p = \frac{3}{32}\cos x + \frac{x}{24}\sin 3x$$

The G.S is $y = y_c + y_p$.

$$y = e^{0x}(c_1\cos 3x + c_2\sin 3x) + \frac{3}{32}\cos x + \frac{x}{24}\sin 3x.$$

Q) Solve $(D^3-1)y = e^x + \sin 3x + 2$.

Sol: G.T, $(D^3-1)y = e^x + \sin 3x + 2$.

which is of the form $f(D)y = Q$

$f(D) = D^3-1$

$Q = e^x + \sin 3x + 2e^{0.x}$

An auxilary eq is $f(m) = 0$.

$m^3 - 1 = 0$

$m = 1$, $m = \frac{-1 \pm \sqrt{3}i}{2}$

$$C.F = y_c = c_1 e^x + e^{-\frac{1}{2}x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$$

$$y.P = P.I = \frac{1}{f(D)}Q$$

$$= \frac{1}{f(D)}\, e^x + \sin 3x + 2e^{0x} \quad \text{--- (1)}$$

$$y.P = \frac{1}{f(D)}e^x = \frac{1}{D^3-1}\, e^x$$

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$$= \frac{1}{(D-1)(D^2+D+1)} e^x = \frac{e^x}{3} \cdot \frac{x^1}{1} \quad \text{--- (2)}$$

$$= \frac{1}{f(D)} \sin 3x = \frac{1}{D^3-1} \sin 3x$$

$$= \frac{1}{D^2 . D-1} \sin 3x = \frac{1}{-3^2 . D-1} = \frac{1}{-9D-1} \sin 3x$$

$$= \frac{\cancel{-9D+1}\,(-9D+1)}{81D^2-1} \sin 3x$$

$$= \frac{(-9D+1) \sin 3x}{81(-3^2)-1} = \frac{-9D+1}{-730} \sin 3x .$$

$$= \frac{1}{730}\left[27 \cos 3x + \sin 3x\right] \quad \text{--- (3)} .$$

$$\frac{1}{D^3-1} 2e^{0x} = 2 . \frac{1}{D^3-1} e^{0x} = \frac{2 . e^{0x}}{-1} \quad \text{--- (4)}$$

Sub (2), (3), (4) in (1)

$$y_p = \frac{e^x . x}{3} + \frac{1}{730}\left[27 (\cos 3x) + \sin 3x\right] + \frac{2e^{0x}}{-1}$$

The G.S is $y = y_c + y_p$

$$y = c_1 e^x + e^{-\frac{1}{2}x}\left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right] + \frac{e^x . x}{2}$$

$$+ \frac{1}{730}\left[27 \cos 3x + \sin 3x\right] + \frac{2e^{0x}}{-1}$$

Q). Solve $(D^2+2D+2)y = e^{-x} + \sin 2x$

Sol: G.T, $(D^2+2D+2)y = e^{-x} + \sin 2x$

which is of the form $f(D)y = Q$

$f(D) = D^2+2D+2$

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$Q = e^{-x} + \sin 2x$

Sln A.E is $f(m) = 0$

$$m^2 + 2m + 2 = 0$$

$$m = -1 \pm i$$

The roots are complex.

$$C.F = y_c = e^{-x}\left[c_1 \cos x + c_2 \sin x\right]$$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$= \frac{1}{D^2+2D+2} e^{-x} + \sin 2x$$

$$y_p = \frac{1}{D^2+2D+2} e^{-x} + \frac{1}{D^2+2D+2} \sin 2x \quad \text{—— (1)}$$

$$\frac{1}{D^2+2D+2} e^{-x} = \frac{1}{(-1)^2+2(-1)+2} e^{-x} = e^{-x} \quad \text{—— (2)}$$

$$\frac{1}{D^2+2D+2} \sin 2x = \frac{1}{-2^2+2D+2} \sin 2x$$

$$= \frac{1}{2D-4} \sin 2x$$

$$= \frac{1}{2} \frac{1}{(D-2)} \sin 2x$$

$$= \frac{1}{2} \frac{D+2}{D^2-4} \sin 2x$$

$$= \frac{1}{2} \frac{D+2}{-2^2-4} \sin 2x$$

$$= \frac{1}{2} \frac{D+2}{-8} \sin 2x$$

$$= \frac{1}{-16} (D+2) \sin 2x$$

$$= \frac{-1}{16} \left[D(\sin 2x + 2\sin 2x\right]$$

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$= \frac{-1}{16}\left[2\cos 2x + 2\sin 2x\right]$

$= \frac{-1}{8}\left[\cos 2x + \sin 2x\right]$

$y_p = e^{-x} - \frac{1}{8}\left[\cos 2x + \sin 2x\right]$

The G.S is , $y = y_c + y_p$

$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{-x} - \frac{1}{8}\left[\cos 2x + \sin 2x\right]$

Type – 3

P.I of $f(D)y = Q$

when $Q = x^k$, where k is a +ve integer.

Consider , D.E of the form $f(D)y = Q$, $Q = x^k$ or a polynomial in x.

$$P.I = y_p = \frac{1}{f(D)} \cdot Q = \frac{1}{f(D)} x^k$$

$$= \frac{1}{[1 \pm \phi(D)]} x^k.$$

To Evaluate P.I we reduce $\frac{1}{f(D)}$ to the form $\frac{1}{1+\phi(D)}$ by taking the lowest degree term from $f(D)$. Now we write $\frac{1}{f(D)}$ as $[1 \pm \phi(D)]^{-1}$

and expand it in ascending powers of D using binomial theorem upto the term containing D^k then operate x^k with the terms of the expansion of $[1 \pm \phi(D)]^{-1}$,

we neglect $D^{k+1}, D^{k+2}, ---$

Since $D^{k+1}(x^k) = 0$

$D^{k+2}(x^k) = 0$

Q) Find the particular integral of $(D^2+3D+2)y = x^3$.

Sol: G.T, $(D^2+3D+2)y = x^3$.

which is of the form $f(D)y = Q$

where $f(D) = D^2+3D+2$.

$Q = x^3$. (Type - 3)

$$P.I = y_p = \frac{1}{f(D)} Q = \frac{1}{(D^2+3D+2)} \cdot x^3 = \frac{1}{2\left[1+\left(\frac{D^2+3D}{2}\right)\right]} x^3$$

$$= \frac{1}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1} x^3$$

W.K.T $(1+x)^{-1} = 1-x+x^2-x^3+ ---, |x|<1$ (we neglect D^4, D^5)

$$= \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2-\left(\frac{D^2+3D}{2}\right)^3 ---\right] x^3$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3}{2}D+\frac{9D^2}{4}+\frac{6}{4}D^3+\frac{27}{8}D^3\right] x^3$$

$$= \frac{1}{2}\left[x^3-\frac{1}{2}D^2(x^3)-\frac{3}{2}D(x^3)+\frac{9}{4}D^2(x^3)+\frac{6}{4}D^3(x^3)-\frac{27}{8}D^3(x^3)\right]$$

$$= \frac{1}{2}\left[x^3-\frac{6x}{2}-\frac{9x^2}{2}+\frac{54x}{4}+\frac{36}{4}-\frac{162}{8}\right]$$

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NOTE: $(1+x)^{-1} = 1-x+x^2-x^3+----, \quad |x|<1$

$$(1-x)^{-1} = 1+x+x^2+x^3+---$$

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+---$$

$$(1+x)^{-2} = 1-2x+3x^2-4x^3+----$$

$$(1+x)^{-3} = 1-3x+6x^2-10x^3+----$$

$$(1-x)^{-3} = 1+3x+6x^2+10x^3+---$$

Q) Solve $(D^3+2D^2+D)y = e^{2x} + \sin 2x + x^2+x.$

Sol: G.T, $(D^3+2D^2+D)y = e^{2x} + \sin 2x + x^2+x.$

which is in the form of $f(D)\, y = Q$

Here $f(D) = D^3+2D^2+D$

$$Q = e^{2x} + \sin 2x + x^2+x.$$

Auxilary eq is $f(m)=0$

$$m^3+2m^2+m=0$$

$$m(m^2+2m+1)=0$$

$$m=0, -1, -1$$

$$C.F = y_c = c_1 e^{0.x} + (c_2 x^0 + c_3 x^1) e^{-x}$$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$y_p = \frac{1}{D^3+2D^2+D}(e^{2x} + \sin 2x + x^2+x)$$

$$y_p = \frac{1}{D^3+2D^2+D} e^{2x} + \frac{1}{D^3+2D^2+D} \sin 2x + \frac{1}{D^3+2D^2+D}(x^2+x) \quad \text{--- (1)}$$

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$$\frac{1}{D^3+2D^2+D}e^{2x} = \frac{1}{18}e^{2x} \quad \text{—— (2)}$$

$f(D) = D^3+2D^2+D \quad , a = 2$

$f(a) = f(2) = 8+8+2 = 18$

$$\frac{1}{D^3+2D^2+D}\sin 2x = \frac{-x}{4}\cos 2x = \frac{1}{D^2 \cdot D + 2D^2 + D}\sin 2x$$

$$= \frac{1}{D(-2)^2+2(-2^2)+D}(\sin 2x) = \frac{1}{-4D-8+D}\sin 2x$$

$$= \frac{-1}{3D+8}\sin 2x = \frac{-(3D-8)}{9D^2-8^2}\sin 2x$$

$$= \frac{8-3D}{9(-2^2)-8^2}\sin 2x = \frac{8-3D}{-100}\sin 2x$$

$$= \frac{3D-8}{100}\sin 2x$$

$$= \frac{1}{100}\left[3D(\sin 2x) - 8\sin 2x\right]$$

$$= \frac{1}{100}\left[(6\cos 2x) - \cancel{16\cos 2x}\ 8\sin 2x\right] \quad \text{—— (3)}$$

$$\frac{1}{D^3+2D^2+D}(x^2+x) = \frac{1}{D\left[1+\left(\frac{D^3+2D^2}{D}\right)\right]}(x^2+x)$$

$$= \frac{1}{D\left[1+(D^2+2D)\right]}(x^2+x)$$

$$= \frac{1}{D}\left[1+(D^2+2D)\right]^{-1}(x^2+x)$$

W.k.T, $(1+x)^{-1} = 1-x+x^2-x^3+\ldots$

Here $x = D^2+2D$.

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$$= \frac{1}{D}\left[1-(D^2+2D)+(D^2+2D)^2\right](x^2+x)$$

$$= \frac{1}{D}\left[1-D^2-2D+4D^2\right](x^2+x)$$

$$= \frac{1}{D}\left[1-2D+3D^2\right](x^2+x)$$

$$= \frac{1}{D}\left[(x^2+x)-2D(x^2+x)+3D^2(x^2+x)\right]$$

$$= \frac{1}{D}\left[(x^2+x)-2(2x+1)+3(2)\right]$$

$$= \frac{1}{D}\left[(x^2+x)-4x-2+6\right] = \frac{1}{D}\left[x^2+x-4x+4\right]$$

$$= \frac{1}{D}\left[x^2-3x+4\right]$$

$$= \int x^2-3x+4\,dx = \frac{x^3}{3}-\frac{3x^2}{2}+4x \quad \text{——④.}$$

Substitute, ②, ③, & ④ in ①, we get

$$y_p = \frac{e^{2x}}{18}+\frac{1}{100}(6\cos 2x-8\sin 2x)+\frac{x^3}{3}-\frac{3x^2}{2}+4x$$

The G.S is $y = y_c + y_p$

$$y = c_1e^{0x}+(c_2x^0+c_3x^1)e^{-x}+\frac{e^{2x}}{18}+\frac{1}{100}(6\cos 2x-8\sin 2x)$$
$$+\frac{x^3}{3}-\frac{3x^2}{2}+4x.$$

Q) Solve $(D^2-2D+4)y = x^2+e^{2x}+\sin 2x$.

Sol: Given, $(D^2-2D+4)y = x^2+e^{2x}+\sin 2x$.

which is of the form, $f(D)y = Q$

$$f(D) = D^2-2D+4$$

$$Q = x^2+e^{2x}+\sin 2x$$

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Auxilary eq is $f(m)=0$

$$m^2-2m+1=0$$

$$m = 1 \pm \sqrt{3}\, i \quad \text{(imaginary roots)}$$

$$C.F = y_c = e^{x}\left[C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right]$$

$$P.I = y_p = \frac{1}{f(D)} Q.$$

$$y_p = \frac{1}{D^2-2D+4}\left(x^2+e^{2x}+\sin 2x\right)$$

$$= \frac{1}{D^2-2D+4}x^2 + \frac{1}{D^2-2D+4}e^{2x} + \frac{1}{D^2-2D+4}\sin 2x. \quad \text{—— (1)}$$

$$\frac{1}{D^2-2D+4}x^2 = \frac{1}{4\left[1+\left(\frac{D^2-2D}{4}\right)\right]}x^2$$

$$= \frac{1}{4}\left[1+\left(\frac{D^2-2D}{4}\right)\right]^{-1}x^2$$

W.K.T $(1+x)^{-1} = 1-x+x^2-x^3+\cdots$

Here $x = \dfrac{D^2-2D}{4}.$

$$= \frac{1}{4}\left[1-\left(\frac{D^2-2D}{4}\right)+\left(\frac{D^2-2D}{4}\right)^2\right]x^2$$

$$= \frac{1}{4}\left[1-\cancel{\frac{D^2}{4}}+\frac{1}{2}D+\cancel{\frac{1}{4}D^2}\right]x^2$$

$$= \frac{1}{4}\left[x^2+\frac{1}{2}D(x^2)\right] = \frac{1}{4}\left[x^2+x\right] \quad \text{—— (2).}$$

$$\frac{1}{D^2-2D+4}e^{2x} = \frac{1}{2^2-2.2+4}e^{2x} = \frac{e^{2x}}{4} \quad \text{—— (3)}$$

$$\frac{1}{D^2-2D+4}\sin 2x = \frac{1}{-2^2-2D+4}\sin 2x = \frac{-1}{2}\frac{1}{D}(\sin 2x)$$

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$= \frac{-1}{2}\int (\sin 2x)\, dx$

$= \frac{\cos 2x}{4}$ — ④.

Sub ②, ③ & ④ in ①

$y_p = \frac{1}{4}(x^2+x) + \frac{e^{2x}}{4} + \frac{\cos 2x}{4}$

The G.S is $y = y_c + y_p$.

$y = e^{x}\left[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right] + \frac{1}{4}(x^2+x) + \frac{e^{2x}}{4} + \frac{\cos 2x}{4}$

Q) Solve. $(D^2+1)^2 y = x^4 + 2\sin x \cos 3x$

Sol: G.T, $(D^2+1)^2 y = x^4 + 2\sin x \cos 3x$

which is of the form, $f(D)y = Q$

$f(D) = (D^2+1)^2$

$Q = x^4 + 2\sin x \cos 3x$

An auxiliary eq is $(m^2+1)^2 = 0$.

$[(m+i)(m-i)]^2 = 0$

$m = \pm i, \pm i$

∴ The roots are imaginary,

$C.F = y_c = e^{0x}\left[(c_1x^0 + c_2x^1)\cos x + (c_3x^0 + c_4x^1)\sin x)\right]$

$P.I = y_p = \frac{1}{(D^2+1)^2}(x^4 + 2\sin x \cos 3x)$

$= \frac{1}{(D^2+1)^2}x^4 + \frac{1}{(D^2+1)^2}\left[\sin(x+3x) + \sin(x-3x)\right]$

$= \frac{1}{(1+D^2)^2}x^4 + \frac{1}{(D^2+1)^2}\sin 4x - \frac{1}{(D^2+1)^2}\sin 2x$

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$$= (1+D^2)^{-2}x^4 + \frac{1}{(D^2+1)^2}\sin 4x - \frac{1}{(D^2+1)^2}\sin 2x.$$

$$= (1-2D^2+3D^4-4D^6+\dots)x^4 + \frac{1}{(-4^2+1)^2}\sin 4x - \frac{1}{(-2^2+1)^2}\sin 2x$$

$$= x^4 - 2D^2(x^4) + 3D^4(x^4) - \frac{1}{225}\sin 4x - \frac{1}{9}\sin 2x$$

$$= x^4 - 24x^2 + 72 - \frac{1}{225}\sin 4x - \frac{1}{9}\sin 2x.$$

The G.S is $y = y_c + y_p$

$$y = e^{0x}\left[(c_1x^0 + c_2x^1)\cos x + (c_3x^0 + c_4x^1)\sin x\right] + x^4 - 24x^2 + 72$$
$$- \frac{1}{225}\sin 4x - \frac{1}{9}\sin 2x.$$

Q) Solve $(D-2)^2y = 8(e^{2x} + \sin 2x + x^2)$

Sol: G.T, $(D-2)^2y = 8(e^{2x} + \sin 2x + x^2)$

which is of the form, $f(D)y = Q$

$f(D) = (D-2)^2$

$Q = 8(e^{2x} + \sin 2x + x^2)$

An auxilary eq is $f(m) = 0$

$(m-2)^2 = 0.$

$m = 2, 2$

$\therefore$ The roots are real & repeat.

C.F = $y_c = (c_1x^0 + c_2x^1)e^{2x}$

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$$P.I = y_p = \frac{1}{f(D)} Q$$

$$y_p = \frac{1}{(D-2)^2} 8(e^{2x} + \sin 2x + x^2)$$

$$= 8 \cdot \frac{1}{(D-2)^2} e^{2x} + 8 \frac{1}{D^2-4D+4} \sin 2x + 8 \cdot \frac{1}{(D-2)^2} \cdot x^2 \quad \text{— (1)}$$

$$\frac{1}{(D-2)^2} e^{2x} = x \cdot \frac{1}{2(D-2)} e^{2x} = x \cdot x \cdot \frac{1}{2(1)} e^{2x} \quad \text{— (2)}$$

$$\frac{1}{D^2-4D+4} \sin 2x = \frac{1}{-\cancel{4}-4D+\cancel{4}} \sin 2x$$

$$= \frac{1}{-4D} \sin 2x = \frac{-1}{4} \frac{1}{D} (\sin 2x)$$

$$= \frac{-1}{4}\left(\frac{-\cos 2x}{2}\right) \quad \text{— (3)}$$

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{4(1-\frac{D}{2})^2} x^2 = \frac{1}{4(1-\frac{D}{2})^2} x^2$$

$$= \frac{1}{4}\left[1 + 2\left(\frac{D}{2}\right) + 3\left(\frac{D^2}{4}\right) + \cdots\right] x^2$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{4} \times 2\right] \quad \text{— (4).}$$

Sub eq 2, 3, 4 in (1)

$$y_p = 8 \cdot x^2 \frac{1}{2} e^{2x} + \cancel{8}\left(\frac{-1}{\cancel{4}}\right)\left(\frac{-\cos 2x}{\cancel{2}}\right) + \frac{\cancel{8}^{2}}{\cancel{4}}\left[x^2 + 2x + \frac{3}{2}\right]$$

$$y_p = 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

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The G.S is $y = y_c + y_p$.

$$y = (c_1 x^0 + c_2 x^1) e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3.$$

8) Solve $(D^2+4)y = e^x + \sin 3x + x^2$.

Sol: G.T, $(D^2+4)y = e^x + \sin 3x + x^2$

which is of the form $f(D)y = Q$

$f(D) = D^2+4$

$Q = e^x + \sin 3x + x^2$

An A.E is $f(m) = 0$

$m^2 + 4 = 0$

$m = \pm 2i$

$$C.F = y_c = e^{0x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

$$P.I = y_p = \frac{1}{f(D)} Q = \frac{1}{D^2+4}(e^x + \sin 3x + x^2)$$

$$y_p = \frac{1}{D^2+4} e^x + \frac{1}{D^2+4} \sin 3x + \frac{1}{D^2+4} x^2 \quad \text{--- (1)}$$

$$\frac{1}{D^2+4} e^x = \frac{1}{1^2+4} e^x = \frac{1}{5} e^x \quad \text{--- (2)}$$

$$\frac{1}{D^2+4} \sin 3x = \frac{1}{-3^2+4} \sin 3x$$

$$= \frac{1}{-5} \sin 3x \quad \text{--- (3)}$$

$$\frac{1}{D^2+4} x^2 = \frac{1}{4\left(1+\frac{D^2}{4}\right)} x^2 = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} x^2$$

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W.K.T, $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$$= \left[1 - \left(1+\frac{D^2}{4}\right) + \left(1+\frac{D^2}{4}\right)^2\right] x^2$$

$$= \left[\cancel{1} - \cancel{1} + 1 + \frac{D^2}{4} + \frac{2D^2}{4}\right] x^2$$

$$= x^2 - \frac{1}{4} D^2(x^2) + \frac{1}{2} D^2(x^2)$$

$$= x^2 - \frac{1}{4}(2) + 1.$$

$$= x^2 + \frac{1}{2}. \quad \text{—④}$$

~~$y_p =$~~ Sub ⑤, ③, ④ in ①.

$$y_p = \frac{1}{5}e^x + \left(\frac{-1}{5}\sin 3x\right) + x^2 + \frac{1}{2}$$

The G.S is,

$$y = y_c + y_p$$

$$y = e^{0x}\left[C_1 \cos 2x + C_2 \sin 2x\right] + \frac{1}{5}e^x - \frac{1}{5}\sin 3x + x^2 + \frac{1}{2}$$

Type - ④

Particular integral of $f(D)y = Q$ when $Q = e^{ax}V$ where 'a' is a Constant and $V = \sin bx$ (or) $\cos bx$ (or) x^k

Consider the D.E of the form $f(D)y = Q$, $Q = e^{ax}V$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$y_p = \frac{1}{f(D)} e^{ax} V.$$

$$y_p = e^{ax} \frac{1}{f(D+a)} V.$$

i) $Q = e^{ax}(\sin bx$ or $\cos bx)$ ~~or x^k~~

First we apply Type-④ and then we apply Type-②

ii) If $Q = e^{ax} x^k$, first we apply Type ④ and then we apply Type -③.

Q) Solve $\frac{d^2y}{dx^2} + y = e^{-x} + x^3 + e^x \sin x.$

Sol: G.T., $\frac{d^2y}{dx^2} + y = e^{-x} + x^3 + e^x \sin x$

The operator form of the given D.E is

$$(D^2+1)y = e^{-x} + x^3 + e^x \sin x,$$

Here $f(D) = D^2+1$

$$Q = e^{-x} + x^3 + e^x \sin x.$$

The auxilary eq. is $f(m) = 0$. i.e. $m^2+1 = 0$.

$$m = \pm i.$$

The roots are imaginary.

$$CF = y_c = C_1 \cos x + C_2 \sin x,$$

$$P.I = y_p = \frac{1}{f(D)} Q.$$

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$$y_p = \frac{1}{D^2+1}\left(e^{-x}+x^3+e^x\sin x\right)$$

$$y_p = \frac{1}{D^2+1}e^{-x} + \frac{1}{D^2+1}x^3 + \frac{1}{D^2+1}e^x\sin x \quad \text{--- (1)}$$

$$\frac{1}{D^2+1}e^{-x} = \frac{1}{(-1)^2+1}e^{-x} = \frac{e^{-x}}{2} \quad \text{--- (2)}$$

$$\frac{1}{D^2+1}x^3 = (1+D^2)^{-1}x^3.$$

W.K.T, $(1+x)^{-1} = 1-x+x^2-x^3+\cdots$

Here, $x = D^2$

$$= [1-D^2+D^4]\,x^3.$$

$$= x^3 - D^2(x^3) + D^4(x^3) = x^3 - 6x \quad \text{--- (3)}$$

$$\frac{1}{D^2+1}e^x\sin x = e^x\frac{1}{(D+1)^2+1}\sin x.$$

W.K.T $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V.$

Here $a=1$, $V=\sin x$.

$$= e^x\frac{1}{D^2+2D+2}\sin x$$

$$= e^x\frac{1}{-1^2+2D+2}\sin x$$

$$= e^x\frac{1}{2D+1}\sin x.$$

$$= e^x\frac{(2D-1)}{(2D+1)(2D-1)}\sin x$$

$$= \frac{e^x(2D-1)}{4D^2-1}\sin x$$

$$= e^x\frac{(2D-1)}{4(-1)-1}\sin x.$$

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$$= \frac{-e^{x}}{5}\left(2D(\sin x) - \sin x\right).$$

$$= \frac{-e^{x}}{5}\left(2\cos x - \sin x\right) \quad \text{——④.}$$

Sub eq ⑤, ③, ④ in ①.

$$y_p = \frac{e^{-x}}{2} + x^3 - 6x - \frac{e^{x}}{5}(2\cos x - \sin x).$$

The G.S is $y = y_c + y_p$.

$$y = c_1\cos x + c_2\sin x + \frac{e^{-x}}{2} + x^3 - 6x - \frac{e^{x}}{5}(2\cos x - \sin x).$$

Q) Solve $(D^2-4)y = x^2\sinh x + \cos 2x + e^{-2x}$.

Sol: G.T, $(D^2-4)y = x^2\sinh x + \cos 2x + e^{-2x}$.

which is of the form $f(D)y = Q$.

Here $f(D) = D^2-4$

$$Q = e^{-2x} + \cos 2x + x^2\sinh x = e^{-2x} + \cos 2x + x^2\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$Q = e^{-2x} + \cos 2x + \frac{1}{2}e^{x}x^2 - \frac{1}{2}e^{-x}x^2.$$

An auxilary eq is $f(m) = 0$. i.e. $m^2 - 4 = 0$

$m = \pm 2$

The roots are real & distinct

$$C.F = y_c = c_1e^{-2x} + c_2e^{2x}.$$

$$P.I = y_p = \frac{1}{f(D)}Q$$

$$y_p = \frac{1}{D^2-4}\left(e^{-2x} + \cos 2x + \frac{1}{2}e^{x}x^2 - \frac{1}{2}e^{-x}x^2\right)$$

$$y_p = \frac{1}{D^2-4}e^{-2x} + \frac{1}{D^2-4}\cos 2x + \frac{1}{D^2-4}\left(\frac{1}{2}e^{x}x^2\right) - \frac{1}{D^2-4}\left(\frac{1}{2}e^{-x}x^2\right) \quad \text{——①}$$

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$$\frac{1}{D^2-4}e^{-2x} = \frac{1}{(D-2)(D+2)}e^{-2x} = \frac{x^1}{1!}\frac{e^{-2x}}{-4} = \frac{-x e^{-2x}}{4} \quad \text{—(2)}$$

$f(D) = D^2-4$, $Q = e^{-2x}$, $a = -2$

$f(a) = f(-2) = 0.$

$f(D) = (D-2)(D+2)$

$(D+2)$ is factor of $f(D)$, $K=1$

$\phi(D) = D-2$, $\phi(a) = \phi(-2) = -4 \neq 0.$

$$\frac{1}{D^2-4}\cos 2x = \frac{1}{-2^2-4}\cos 2x = -\frac{\cos 2x}{8} \quad \text{—(3)}$$

$$\frac{1}{D^2-4}e^x x^2 = e^x \frac{1}{(D+1)^2-4}x^2.$$

W.K.T $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V.$

$$= e^x \frac{1}{D^2+2D-3}x^2.$$

$$= e^x \frac{1}{-3\left[1-\left(\frac{D^2+2D}{3}\right)\right]}x^2.$$

$$= \frac{e^x}{-3}\left[1-\left(\frac{D^2+2D}{3}\right)\right]^{-1}x^2.$$

W.K.T, $(1-x)^{-1} = 1+x+x^2+x^3+\ldots$

$$= \frac{e^x}{-3}\left[1+\left(\frac{D^2+2D}{3}\right)+\left(\frac{D^2+2D}{3}\right)^2\right]x^2$$

$$= \frac{e^x}{-3}\left[1+\frac{D^2}{3}+\frac{2}{3}D+\frac{4}{9}D^2\right]x^2.$$

$$= \frac{-e^x}{3}\left[x^2+\frac{1}{3}D^2(x^2)+\frac{2}{3}D(x^2)+\frac{4}{9}D^2(x^2)\right]$$

$$= \frac{-e^{+x}}{3}\left[x^2+\frac{2}{3}+\frac{4x}{3}+\frac{8}{9}\right] \quad \text{—(4)}$$

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$$\frac{1}{D^2-4} e^{-x} x^2 = e^{-x} \frac{1}{(D-1)^2-4} . x^2.$$

$$= e^{-x} \frac{1}{D^2-2D-3} x^2.$$

$$= e^{-x} \frac{1}{(-3)\left[1-\left(\frac{D^2-2D}{3}\right)\right]} x^2.$$

$$= -\frac{e^{-x}}{3} \left[1-\left(\frac{D^2-2D}{3}\right)\right]^{-1} x^2$$

W.K.T $(1-x)^{-1} = 1+x+x^2+ \text{----}$

$$= -\frac{e^{-x}}{3}\left[1+\left(\frac{D^2-2D}{3}\right)+\left(\frac{D^2-2D}{3}\right)^2\right] x^2.$$

$$= -\frac{e^{-x}}{3}\left[1+\frac{D^2}{3}-\frac{2}{3}D+\frac{4}{9}D^2\right] x^2.$$

$$= -\frac{e^{-x}}{3}\left[x^2+\frac{1}{3}D^2(x^2)-\frac{2}{3}D(x^2)+\frac{4}{9}D^2(x^2)\right]$$

$$= -\frac{e^{-x}}{3}\left[x^2+\frac{2}{3}-\frac{4x}{3}+\frac{8}{9}\right] \quad \text{—} (5)$$

Sub 2, 3, 4, 5 in 1

$$y_p = \frac{-xe^{-2x}}{4} - \frac{\cos 2x}{8} - \frac{e^{x}}{6}\left(x^2+\frac{2}{3}+\frac{4x}{3}+\frac{8}{9}\right)$$
$$+ \frac{e^{-x}}{6}\left(x^2+\frac{2}{3}-\frac{4x}{3}+\frac{8}{9}\right)$$

The G.S is,

$y = y_c + y_p$

$$y = c_1 e^{-2x} + c_2 e^{2x} - \frac{xe^{-2x}}{4} - \frac{\cos 2x}{8} - \frac{e^{x}}{6}\left(x^2+\frac{2}{3}+\frac{4x}{3}+\frac{8}{9}\right)$$
$$+ \frac{e^{-x}}{6}\left(x^2+\frac{2}{3}-\frac{4x}{3}+\frac{8}{9}\right).$$

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Q) Solve $(D^2-4D+3)y = xe^{3x} + e^{x}\cos x + e^{x} + \sin x$.

Sol: G.T, $(D^2-4D+3)y = xe^{3x} + e^{x}\cos x + e^{x} + \sin x$

which is of the form $f(D)y = Q$

$f(D) = D^2-4D+3$

$Q = e^{x} + xe^{3x} + e^{x}\cos x + \sin x$.

The auxiliary eq is $f(m)=0$.

$m^2-4m+3=0$

$m = 1, 3$.

The roots are real & distinct

$C.F = y_c = c_1e^{x} + c_2e^{3x}$

$P.I = y_p = \frac{1}{f(D)}Q$

$y_p = \frac{1}{D^2-4D+3}(e^{x} + xe^{3x} + e^{x}\cos x + \sin x)$

$y_p = \frac{1}{D^2-4D+3}e^{x} + \frac{1}{D^2-4D+3}xe^{3x} + \frac{1}{D^2-4D+3}e^{x}\cos x + \frac{\sin x}{D^2-4D+3}$ —— (I)

$\frac{1}{D^2-4D+3}xe^{3x} =$

W.K.T, $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$

$= e^{3x}\frac{1}{(D+3)^2-4(D+3)+3}x$

$= e^{3x}\frac{1}{D^2+9+6D-4D-12+3}x$

$= e^{3x}\frac{1}{D^2+2D}x$.

$= e^{3x}\frac{1}{2D(1+\frac{D}{2})}x = \cancel{\frac{e^{3x}}{x}\int} = e^{3x}\frac{1}{2D}\left[1+\frac{D}{2}\right]^{-1}x$

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W.K.T, $(1+x)^{-1} = 1-x+x^2-x^3+\dots$

$= e^{3x}\frac{1}{2D}\left(1-\frac{D}{2}\right)x \quad \cancel{= \frac{e^{3x}}{2}\int \frac{-D}{2}x}$

$\cancel{= \frac{-e^{3x}}{4}\int x+D} \quad = \cancel{-\frac{e^{3x}\cdot x^2}{8}}$ — (2)

$= \frac{e^{3x}}{2}\int\left(x-\frac{Dx}{2}\right)dx$

$= \frac{e^{3x}}{2}\left[\frac{x^2}{2}-\frac{x}{2}\right]$ — (2)

$\frac{1}{D^2-4D+3}e^x\cos 2x =$

W.K.T, $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$. $\quad a=1,\ V=\cos 2x$

$= e^x\frac{1}{D^2-4D+3}\cos 2x \quad = e^x\frac{1}{(D+1)^2-4(D+1)+3}\cos 2x$

$= e^x\frac{1}{-4D-1}\cos 2x \quad = e^x\frac{1}{D^2-2D}\cos 2x$

$= e^x\frac{1}{-2D-4}\cos 2x = e^x\frac{1}{-2(D+2)}\cos 2x$

$= \frac{e^x}{-2}\frac{D-2}{D^2-4}\cos 2x \quad = \frac{e^x}{-2}\frac{D-2}{-8}\cos 2x$

$= \frac{e^x}{16}\left[-2\sin 2x - 2\cos 2x\right]$ — (3)

$= \frac{-e^x}{8}\left[\sin 2x+\cos 2x\right]$

$\frac{1}{D^2-4D+3}e^x = x\cdot\frac{1}{f'(D)}e^{ax}$ when $f(a)=0$.

$= x\cdot\frac{1}{2D-4}e^x = \frac{x}{-2}e^x$ — (4).

$\frac{1}{D^2-4D+3}\sin x = \frac{1}{-4D+2}\sin x \quad = \frac{1}{-2(2D-1)}\sin x$

$= \frac{-1}{2}\frac{2D+1}{4D^2-1}\sin x \quad = \frac{-1}{2}\frac{(2D+1)}{-5}\sin x$

$= \frac{1}{10}(2D+1)\sin x$

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$$= \frac{1}{10}\left[2 D(\sin x) + \sin x\right]$$

$$= \frac{1}{10}\left[2\cos x + \sin x\right] \quad \text{—— (5)}$$

Sub 2, 3, 4, 5 in (1)

$$y_p = \frac{-x}{2}e^{x} + \frac{e^{3x}}{2}\left[\frac{x^2}{2} - \frac{x}{2}\right] \cancel{\frac{x}{2}e^{x}} + \frac{1}{10}\left[2\cos x + \sin x\right] - \frac{e^{x}}{8}\left[\sin 2x + \cos 2x\right]$$

The G.S is $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{3x} - \frac{x}{2}e^{x} + \frac{e^{3x}}{2}\left[\frac{x^2}{2} - \frac{x}{2}\right] + \frac{1}{10}\left[2\cos x + \sin x\right] - \frac{e^{x}}{8}\left[\sin 2x + \cos 2x\right]$$

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Type - 5

Particular integral of $f(D)y = Q$ when $Q = x^m V$ where m is positive integer and $V = \sin bx$ or $\cos bx$.

Consider the D.E of the form $f(D)y = Q$ where $Q = x^m V$.

Case - 1 : when $m = 1$, $Q = xV$.

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$y_p = \frac{1}{f(D)} xV.$$

$$= \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} V.$$

Q) Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x \sin x + x^2 + e^{-x}$

Sol: G.T, $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x \sin x + x^2 + e^{-x}$.

An operator form of the given D.E is $f(D)y = Q$

i.e. $(D^2+3D+2)y = xe^x \sin x + e^{-x} + x^2$.

$$f(D) = D^2+3D+2$$

$$Q = e^{-x} + x^2 + xe^x \sin x.$$

An auxilary eq is $f(m) = 0$.

i.e. $m^2+3m+2 = 0$.

$m = -1, -2$

The roots are real & distinct.

$$C.F = y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$P.I = y_p = \frac{1}{f(D)} Q.$$

$$y_p = \frac{1}{D^2+3D+2}(e^{-x} + x^2 + e^x x \sin x)$$

$$y_p = \frac{1}{D^2+3D+2} e^{-x} + \frac{1}{D^2+3D+2} x^2 + \frac{1}{D^2+3D+2} e^x x \sin x \quad \text{---} (1)$$

$$\frac{1}{D^2+3D+2} e^{-x} = \frac{1}{(D+1)(D+2)} e^{-x} = \frac{x^1}{1!} \frac{e^{-x}}{1} = x e^{-x} \quad \text{---} (2)$$

$$\frac{1}{D^2+3D+2} x^2 = \frac{1}{2\left[1+\left(\frac{D^2+3D}{2}\right)\right]} x^2$$

$$= \frac{1}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1} x^2$$

W.K.T, $(1+x)^{-1} = 1-x+x^2-x^3+ \cdots$

$$= \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\right] x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3}{2}D+\frac{9D^2}{4}\right] x^2$$

$$= \frac{1}{2}\left[x^2 - \frac{1}{2}D^2(x^2) - \frac{3}{2}D(x^2) + \frac{9}{4}D^2(x^2)\right]$$

$$= \frac{1}{2}\left[x^2 - 1 - 3x + \frac{9}{2}\right] \quad \text{---} (3)$$

$$\frac{1}{D^2+3D+2} e^x x \sin x = e^x \frac{1}{(D+1)^2+3(D+1)+2} x \sin x$$

W.K.T, $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$.

$$= e^x \frac{1}{D^2+5D+6} x \sin x$$

W.K.T $\frac{1}{f(D)} xV = \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} V$.

$$= e^x \left[x - \frac{2D+5}{D^2+5D+6}\right] \frac{1}{D^2+5D+6} \sin x.$$

$$= e^x \left[x - \frac{2D+5}{D^2+5D+6}\right] \frac{1}{-1^2+5D+6} \sin x.$$

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$$= \frac{e^x}{15}\left[x - \frac{2D+5}{D^2+5D+6}\right]\frac{1}{D+1}\sin x.$$

$$= \frac{e^x}{5}\left[x - \frac{2D+5}{D^2+5D+6}\right]\frac{D-1}{(D+1)(D-1)}\sin x.$$

$$= \frac{e^x}{5}\left[x - \frac{2D+5}{D^2+5D+6}\right]\frac{D-1}{D^2-1}\sin x.$$

$$= \frac{e^x}{5}\left[x - \frac{2D+5}{D^2+5D+6}\right]\frac{(D-1)}{-1^2-1}\sin x.$$

$$= \frac{e^x}{10}\left[x - \frac{2D+5}{D^2+5D+6}\right](1-D)\sin x$$

$$= \frac{e^x}{10}\left[x - \frac{2D+5}{D^2+5D+6}\right][\sin x - D(\sin x)]$$

$$= \frac{e^x}{10}\left[x - \frac{2D+5}{D^2+5D+6}\right](\sin x - \cos x)$$

$$= \frac{e^x}{10}\left[x(\sin x - \cos x) - \frac{(2D+5)}{(D^2+5D+6)}(\sin x - \cos x)\right]$$

$$(2D+5)(\sin x - \cos x) = 2D(\sin x - \cos x) + 5(\sin x - \cos x)$$

$$= 2(\cos x + \sin x) + 5(\sin x - \cos x)$$

$$= 7\sin x - 3\cos x.$$

$$= \frac{e^x}{10}\left[x(\sin x - \cos x) - \frac{1}{D^2+5D+6}(7\sin x - 3\cos x)\right]$$

$$= \frac{e^x}{10}\left[x(\sin x - \cos x) - \frac{1}{-1^2+5D+6}(7\sin x - 3\cos x)\right]$$

$$= \frac{e^x}{10}\left[x(\sin x - \cos x) - \frac{1}{5(D+1)}(7\sin x - 3\cos x)\right]$$

$$= \frac{e^x}{10}\left[x(\sin x - \cos x) - \frac{D-1}{5(D^2-1)}(7\sin x - 3\cos x)\right]$$

$$= \frac{e^x}{10}\left[x(\sin x - \cos x) - \frac{D-1}{5(-1^2-1)}(7\sin x - 3\cos x)\right]$$

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$$= \frac{e^x}{10}\left[x(\sin x - \cos x) + \frac{1}{10}(D-1)(7\sin x - 3\cos x)\right]$$

$$= \frac{e^x}{10}\left[x(\sin x - \cos x) + \frac{1}{10}\left[D(7\sin x - 3\cos x) - 7\sin x + 3\cos x\right]\right]$$

$$= \frac{e^x}{10}\left[x(\sin x - \cos x) + \frac{1}{10}(7\cos x + 3\sin x - 7\sin x + 3\cos x)\right]$$

$$= \frac{e^x}{10}\left[x(\sin x - \cos x) + \frac{1}{10}(10\cos x - 4\sin x)\right] \quad — ④$$

Sub ②, ③, ④ in ①, we get.

$$y_p = xe^{-x} + \frac{1}{2}\left[x^2 - 1 - 3x + \frac{9}{2}\right] + \frac{xe^x}{10}(\sin x - \cos x)$$
$$+ \frac{e^x}{100}(10\cos x - 4\sin x).$$

The G.S is $y = y_c + y_p$

$$y = c_1 e^{-x} + c_2 e^{-2x} + xe^{-x} + \frac{1}{2}\left(x^2 - 1 - 3x + \frac{9}{2}\right) + \frac{xe^x}{10}(\sin x - \cos x)$$
$$+ \frac{e^x}{100}(10\cos x - 4\sin x).$$

Q) Solve $(D^2 + 5D + 6)y = e^{-2x} + x^2 + xe^x \cos x$

Sol: G.T, $(D^2 + 5D + 6)y = e^{-2x} + x^2 + xe^x \cos x$

which is in the form, $f(D)y = Q$

$f(D) = D^2 + 5D + 6$

$Q = e^{-2x} + x^2 + e^x x\cos x$

An auxilary eq is $f(m) = 0$

$m^2 + 5m + 6 = 0$

$m = -2, -3$

The roots are real & distinct

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$$C.F = y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$= \frac{1}{D^2+5D+6}\left(e^{-2x} + x^2 + e^{x} x \cos x\right)$$

$$= \frac{1}{D^2+5D+6} e^{-2x} + \frac{1}{D^2+5D+6} x^2 + \frac{1}{D^2+5D+6} e^{x} x \cos x \quad \text{—} (1)$$

$$\frac{1}{D^2+5D+6} e^{-2x} = \frac{1}{(D+2)(D+3)} e^{-2x} = \frac{x^1}{1!} e^{-2x} = x e^{-2x} \quad \text{—} (2)$$

$$\frac{1}{D^2+5D+6} x^2 = \frac{1}{6\left[1+\frac{D^2+5D}{6}\right]} x^2 = \frac{1}{6}\left[1+\left(\frac{D^2+5D}{6}\right)\right]^{-1} x^2$$

W.K.T, $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$$= \frac{1}{6}\left[1 - \left(\frac{D^2+5D}{6}\right) + \left(\frac{D^2+5D}{6}\right)^2\right] x^2$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5}{6}(D) + \frac{D^4}{36} + \frac{25}{36} D^2 + \frac{5D^3}{3}\right] x^2$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{6} D^2(x^2) + \frac{5}{6} D(x^2) + \frac{25}{36} D^2(x^2)\right]$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{3} + \frac{5x}{3} + \frac{25}{18}\right] \quad \text{—} (3)$$

$$\frac{1}{D^2+5D+6} e^{x} x \cos x = e^{x} \frac{1}{(D+1)^2 + 5(D+1) + 6} x \cos x$$

W.K.T, $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V.$

$$= e^{x} \frac{1}{D^2+7D+12} x \cos x$$

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W.K.T, $\frac{1}{f(D)} xV = \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} V$

$$= e^x\left[x - \frac{2D+7}{D^2+7D+12}\right]\frac{1}{D^2+7D+12}\cos x$$

$$= e^x\left[x - \frac{2D+7}{D^2+7D+12}\right]\frac{1}{7D+11}\cos x$$

$$= e^x\left[x - \frac{2D+7}{D^2+7D+12}\right]\frac{7D-11}{49D^2-121}\cos x$$

$$= e^x\left[x - \frac{2D+7}{D^2+7D+12}\right]\frac{7D-11}{-170}\cos x$$

$$= \frac{e^x}{-170}\left[x - \frac{2D+7}{D^2+7D+12}\right]\left(7D(\cos x) - 11\cos x\right)$$

$$= \frac{e^x}{-170}\left[x - \frac{2D+7}{D^2+7D+12}\right]\left(-7\sin x - 11\cos x\right)$$

$$= \frac{e^x}{170}\left[x(7\sin x + 11\cos x) - \frac{2D+7}{D^2+7D+12}(7\sin x + 11\cos x)\right]$$

$$(2D+7)(7\sin x + 11\cos x) = 2D(7\sin x + 11\cos x) + 7(7\sin x + 11\cos x)$$

$$= 14\cos x - 22\sin x + 49\sin x + 77\cos x$$

$$= 91\cos x + 27\sin x.$$

$$= \frac{e^x}{170}\left[x(7\sin x + 11\cos x) - \frac{1}{D^2+7D+12}(27\sin x + 91\cos x)\right]$$

$$= \frac{e^x}{170}\left[x(7\sin x + 11\cos x) - \frac{1}{7D+11}(27\sin x + 91\cos x)\right]$$

$$= \frac{e^x}{170}\left[x(7\sin x + 11\cos x - \frac{7D-11}{49D^2-121}(27\sin x + 91\cos x)\right]$$

$$= \frac{e^x}{170}\left[x(7\sin x + 11\cos x) - \frac{7D-11}{-170}(27\sin x + 91\cos x)\right]$$

$(7D-11)(27\sin x+91\cos x) =$

$7D(27\sin x + 91\cos x) - 11(27\sin x+91\cos x)$

$= 189\cos x - 637\sin x - 297\sin x - 1001\cos x$

$= -812\cos x - 934\sin x.$

$$= \frac{e^x}{170}\left[x(7\sin x + 11\cos x) - \frac{1}{170}(812\cos x + 934\sin x)\right] \quad \text{--- } (4)$$

Sub (2), (3), (4) in (1).

$$y_p = xe^{-2x} + \frac{1}{6}\left[x^2 - \frac{1}{3} + \frac{5x}{3} + \frac{25}{18}\right] +$$

$$\frac{e^x}{170}\left[x(7\sin x + 11\cos x) - \frac{1}{170}(812\cos x + 934\sin x)\right]$$

The G.S is $y = y_c + y_p$.

$$y = c_1e^{-2x} + c_2e^{-3x} + xe^{-2x} + \frac{1}{6}\left[x^2 - \frac{1}{3} + \frac{5x}{3} + \frac{25}{18}\right]$$

$$+ \frac{e^x}{170}\left[x(7\sin x + 11\cos x) - \frac{1}{170}(812\cos x + 934\sin x)\right]$$

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Q) Solve $\frac{d^2y}{dx^2} + 4y = x\sin x + e^{-2x} + e^{x}(x^2)$

Sol: G.T, $\frac{d^2y}{dx^2} + 4y = x\sin x + e^{-2x} + e^{x}x^2$.

The operator form of the given D.E is $f(D)y = Q$

i.e $(D^2+4)y = x\sin x + e^{-2x} + e^{x}x^2$.

Here $f(D) = D^2+4$

$Q = x\sin x + e^{-2x} + e^{x}x^2$

The auxiliary eq is $f(m) = 0$.

$m^2+4 = 0$

$m = \pm 2i$

$CF = y_c = (c_1\cos 2x + c_2\sin 2x)e^{0x}$

$P.I = y_p = \frac{1}{f(D)} Q$

$= \frac{1}{D^2+4}(x\sin x + e^{-2x} + e^{x}x^2)$

$y_p = \frac{1}{D^2+4} x\sin x + \frac{1}{D^2+4} e^{-2x} + \frac{1}{D^2+4} e^{x}x^2$. ——①

$\frac{1}{D^2+4} \cancel{x\sin x}\, e^{-2x} = \frac{1}{(-2)^2+4} e^{-2x} = \frac{1}{8} e^{-2x}$ ——②.

$\frac{1}{D^2+4} e^{x}x^2 = e^{x} \frac{1}{(D+1)^2+4} x^2$

W.K.T, $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$.

$= e^{x} \frac{1}{D^2+1+2D+4} x^2 = e^{x} \frac{1}{D^2+2D+5} x^2$

$= e^{x} \frac{1}{5\left[1+\left(\frac{D^2+2D}{5}\right)\right]} x^2 = \frac{e^{x}}{5}\left[1+\left(\frac{D^2+2D}{5}\right)\right]^{-1} x^2$.

W.K.T, $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$$= \frac{e^x}{5}\left[1 - \left(\frac{D^2+2D}{5}\right) + \left(\frac{D^2+2D}{5}\right)^2\right]x^2$$

$$= \frac{e^x}{5}\left[1 - \frac{D^2}{5} - \frac{2}{5}D + \frac{4D^2}{25}\right]x^2$$

$$= \frac{e^x}{5}\left[x^2 - \frac{1}{5}D^2(x^2) + \frac{2}{5}D(x^2) + \frac{4}{25}D^2(x^2)\right]$$

$$= \frac{e^x}{5}\left[x^2 - \frac{1}{5}(2) + \frac{2}{5}(2x) + \frac{4}{25}(2)\right]$$

$$= \frac{e^x}{5}\left(x^2 - \frac{2}{5} + \frac{4x}{5} + \frac{8}{25}\right) \quad \text{--- (3)}$$

$$\frac{1}{D^2+4} x\sin x =$$

W.K.T, $\frac{1}{f(D)} xV = \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} V.$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{D^2+4} \sin x.$$

$$= \left[x - \frac{2D}{D^2+4}\right] \cdot \frac{1}{-1^2+4} \sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{3} \sin x$$

$$= \frac{1}{3} x\sin x - \frac{2}{3} \frac{D(\sin x)}{D^2+4}$$

$$= \frac{1}{3} x\sin x - \frac{2}{3} \frac{\cos x}{D^2+4}$$

$$= \frac{1}{3}\left[x\sin x - \frac{2}{D^2+4}\cos x\right]$$

$$= \frac{1}{3}\left[x\sin x - \frac{2}{-1^2+4}\cos x\right]$$

$$= \frac{1}{3}\left[x\sin x - \frac{2}{3}\cos x\right] \text{ --- (4)}$$

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Sub ②, ③ & ④ in ①

$$y_p = \frac{1}{3}\left[x\sin x - \frac{2}{3}\cos x\right] + \frac{1}{5}e^{-2x} + \frac{e^x}{5}\left(x^2 - \frac{2}{5} + \frac{4x}{5} + \frac{8}{25}\right)$$

The G.S is $y = y_c + y_c$

$$y = e^{0x}(c_1\cos 2x + c_2\sin 2x) + \frac{1}{3}\left[x\sin x - \frac{2}{3}\cos x\right] + \frac{1}{5}e^{-2x}$$
$$+ \frac{e^x}{5}\left(x^2 - \frac{2}{5} + \frac{4x}{5} + \frac{8}{25}\right)$$

Q) Solve $\frac{d^2y}{dx^2} - y = x\cos x + x^2e^x + e^{-x}$.

Sol: G.T, $\frac{d^2y}{dx^2} - y = x\cos x + x^2e^x + e^{-x}$.

An operator form of given D.E is $f(D)y = Q$.

i.e. $(D^2-1)y = x\cos x + x^2e^x + e^{-x}$.

Here, $f(D) = D^2-1$

$Q = x\cos x + x^2e^x + e^{-x}$.

An A.E is $f(m) = 0$.

$m^2 - 1 = 0$

$m = \pm 1$

$C.F = y_c = c_1e^x + c_2e^{-x}$.

$$P.I. = y_p = \frac{1}{f(D)}Q$$

$$= \frac{1}{D^2-1}(x\cos x + x^2e^x + e^{-x})$$

$$y_p = \frac{1}{D^2-1}(x\cos x) + \frac{1}{D^2-1}x^2e^x + \frac{1}{D^2-1}e^{-x} \quad \text{—①}$$

$$\frac{1}{D^2-1}e^{-x} = \frac{x}{1}\cdot\frac{e^{-x}}{-2} \quad \text{—②}$$

$$\frac{1}{D^2-1} x^2 e^x = e^x \frac{1}{(D+1)^2 - 1} x^2$$

W.K.T, $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V.$

$$= e^x \frac{1}{D^2+1+2D-1} x^2$$

$$= e^x \frac{1}{D^2+2D} x^2 = e^x \frac{1}{2D\left[1+\frac{D}{2}\right]} x^2 = \frac{e^x}{2} \frac{1}{D} \left[1+\frac{D}{2}\right]^{-1} x^2.$$

$$= \frac{e^x}{2} \frac{1}{D} \cdot \left[1 - \frac{D}{2} + \frac{D^2}{4}\right] x^2$$

$$= \frac{e^x}{2} \frac{1}{D} \left[x^2 - \frac{1}{2} D(x^2) + \frac{1}{4} D^2(x^2)\right]$$

$$= \frac{e^x}{2} \frac{1}{D} \left[x^2 - \frac{1}{2}(2x) + \frac{1(2)}{4}\right] = \frac{e^x}{2} \frac{1}{D} \left[x^2 - x + \frac{1}{2}\right]$$

$$= \frac{e^x}{2} \left[\int x^2 - \int x + \frac{1}{2}\int 1\right] dx$$

$$= \frac{e^x}{2} \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right] \quad \text{--- (4)}.$$

$$\frac{1}{D^2-1} x \cos x.$$

W.K.T, $\frac{1}{f(D)} x V = \left[x - \frac{f'(D)}{f(D)}\right] \frac{1}{f(D)} V.$

$$= \left[x - \frac{2D}{D^2-1}\right] \frac{1}{D^2-1} \cos x$$

$$= \left[x - \frac{2D}{D^2-1}\right] \frac{1}{-1^2-1} \cos x$$

$$= \left[x - \frac{2D}{D^2-1}\right] \frac{1}{-2} \cos x$$

$$= \frac{-1}{2} x \cos x + \frac{2D}{D^2-1} \cos x$$

$= -\frac{1}{2}x\cos x + \frac{\cancel{2}D}{\cancel{2}(D^2-1)}\cos x$

$= -\frac{1}{2}x\cos x + \frac{D}{D^2-1}\cos x$

$= -\frac{1}{2}x\cos x + \frac{(-\sin x)}{D^2-1}$

$= -\frac{1}{2}x\cos x - \frac{1}{D^2-1}\sin x$

$= -\frac{1}{2}x\cos x - \frac{1}{-1^2-1}\sin x$

$= -\frac{1}{2}x\cos x - \frac{1}{-2}\sin x$

$= -\frac{1}{2}x\cos x + \frac{1}{2}\sin x$ —— (5)

Sub all these in eq (1).

$y_p = \frac{1}{2}\sin x - \frac{1}{2}x\cos x + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right] - x\frac{e^{-x}}{2}$

The G.S is $y = y_c + y_p$

$y = c_1e^x + c_2e^{-x} + \frac{1}{2}\sin x - \frac{1}{2}x\cos x + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right]$
$- x\frac{e^{-x}}{2}$

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Method - ⑤ → Case - ii :-

when $m>1$

Consider the D.E of the form $f(D)y = Q$, $Q = x^m \cdot V$

where V = Sinbx or Cosbx .

W.K.T , $e^{ibx} = \cos bx + i\sin bx$

Real part $(e^{ibx}) = \cos bx$

I.P $(e^{ibx}) = \sin bx$

i) $Q = x^m \sin bx$

$$P.I = y_p = \frac{1}{f(D)} Q = \frac{1}{f(D)} x^m \sin bx$$

$$= \frac{1}{f(D)} x^m \, I.P(e^{ibx})$$

$$= I.P \frac{1}{f(D)} x^m e^{ibx}.$$

ii) $Q = x^m \cos bx$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$y_p = \frac{1}{f(D)} x^m \cos bx$$

$$= \frac{1}{f(D)} x^m \, R.P(e^{ibx})$$

$$= R.P \frac{1}{f(D)} x^m e^{ibx}.$$

Here first we apply method ④ and then we apply method ③

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*Q) Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x + \cos x + e^{2x}$

Sol: G.T, $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x + \cos x + e^{2x}$

An eq of the form $f(D)y = Q$

$f(D) = D^2-4D+4$

$Q = 8x^2e^{2x}\sin 2x + \cos x + e^{2x}$

An A.E is $f(m) = 0$

$m^2-4m+4 = 0$

$m = 2, 2$.

The roots are real & repeated.

$C.F = y_c = (c_1+c_2x)e^{2x}$

$P.I = y_p = \frac{1}{f(D)}Q$

$y_p = \frac{1}{D^2-4D+4}\left[8x^2e^{2x}\sin 2x + \cos x\right] + e^{2x}$

$y_p = \frac{1}{D^2-4D+4}\cos x + \frac{1}{D^2-4D+4}8x^2e^{2x}\sin 2x + \frac{1}{D^2-4D+4}e^{2x}$ —— (1)

$\frac{1}{D^2-4D+4}\cos x = \frac{1}{-1^2-4D+4}\cos x$

~~W.K.T~~ $= \frac{1}{-4D+3}\cos x = \frac{3+4D}{9-16D^2}\cos x$

$= \frac{3+4D}{9-16(-1^2)}\cos x = \frac{3+4D}{25}\cos x$

$= \frac{1}{25}\left[3\cos x + 4D\cos x\right]$

$= \frac{1}{25}\left[3\cos x - 4\sin x\right]$ —— (2)

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$$\frac{1}{\cancel{D^2-4D+4}}\ \cancel{8e^{2x}x^2\sin 2x}$$

$$\frac{1}{D^2-4D+4}e^{2x}=\frac{1}{(D-2)^2}e^{2x}=\frac{x^2}{2!}\frac{e^{2x}}{1}=x^2e^{2x} \quad \text{——} (3)$$

$$\frac{1}{D^2-4D+4}8x^2e^{2x}\sin 2x=8\cdot\frac{1}{(D-2)^2}e^{2x}x^2\sin 2x.$$

W.K.T, $\frac{1}{f(D)}e^{ax}V=e^{ax}\frac{1}{f(D+a)}V.$

$$=8e^{2x}\frac{1}{(D+2-2)^2}x^2\sin 2x.$$

$$=8e^{2x}\frac{1}{D^2}x^2\sin 2x.$$

$$=8e^{2x}\,I.P\,\frac{1}{D^2}e^{i2x}x^2.$$

$$=8e^{2x}\,I.P\,e^{2ix}\frac{1}{(D+2i)^2}x^2.$$

$$=8e^{2x}\,I.P\,e^{2ix}\frac{1}{(2i)^2\left[1+\frac{D}{2i}\right]^2}x^2.$$

$$=-2e^{2x}\,I.P\text{ of }e^{2ix}\left(1+\frac{D}{2i}\right)^{-2}x^2$$

W.K.T, $(1+x)^{-2}=1-2x+3x^2-4x^3+----$

$$=-2e^{2x}\,I.P\text{ of }e^{2ix}\left[1-2\left(\frac{D}{2i}\right)+3\left(\frac{D^2}{4i^2}\right)\right]x^2$$

$$=-2e^{2x}\,I.P\text{ of }e^{2ix}\left[x^2+iD(x^2)-\frac{3}{4}D^2(x^2)\right]$$

$$=-2e^{2x}\,I.P\text{ of }e^{2ix}\left[\left(x^2-\frac{3}{2}\right)+i2x\right]$$

$$=-2e^{2x}\,I.P\text{ of }(\cos 2x+i\sin 2x)\left[\left(x^2-\frac{3}{2}\right)+i2x\right]$$

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$= -2e^{2x}\, I.P \text{ of } \left[\left(x^2-\frac{3}{2}\right)\cos 2x - 2x\sin 2x\right] + i\left[2x\cos 2x + \left(x^2-\frac{3}{2}\right)\sin\right.$

$= -2e^{2x}\left[(2x\cos 2x) + \left(x^2-\frac{3}{2}\right)\sin 2x\right]$ ---- (4)

$$y_p = \frac{1}{25}\left[3\cos x - 4\sin x\right] + \frac{x^2e^{2x}}{2} - 2e^{2x}\left[2x\cos x + \left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

The G.S is $y = y_c + y_p$

$$y = (c_1 + c_2x)e^{2x} + \frac{1}{25}\left[3\cos x - 4\sin x\right] + \frac{x^2e^{2x}}{2}$$
$$- 2e^{2x}\left[2x\cos x + \left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

Q) Solve $(D^4+2D^2+1)y = x^2\cos^2 x + e^{-x} + x^3$.

Sol: G.T, $(D^4+2D^2+1)y = x^2\cos^2 x + e^{-x} + x^3$

The eq is of the form $f(D)y = Q$

$f(D) = D^4 + 2D^2 + 1$

$Q = x^2\cos^2 x + e^{-x} + x^3$

The A.E is $f(m) = 0$.

$m^4 + 2m^2 + 1 = 0$.

$(m^2+1)^2 = 0$.

$[(m+i)(m-i)]^2 = 0$

$m = \pm i, \pm i$.

∴ Roots are imaginary.

$C.F = y_c = e^{0x}\left[(c_1x^0 + c_2x^1)\cos x + (c_3x^0 + c_4x^1)\sin x\right]$

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$P.I = y_P = \frac{1}{f(D)} Q$

$$y_P = \frac{1}{D^4+2D^2+1}\left(x^2\cos^2 x + e^{-x} + x^3\right)$$

$$y_P = \frac{1}{D^4+2D^2+1} x^2\cos^2 x + \frac{1}{D^4+2D^2+1} e^{-x} + \frac{1}{D^4+2D^2+1} x^3 \quad \text{— (1)}$$

$$\frac{1}{D^4+2D^2+1} e^{-x} = \frac{1}{(-1)^4+2(-1)^2+1} e^{-x} = \frac{1}{1+2+1} e^{-x} = \frac{1}{4} e^{-x} \quad \text{— (2)}$$

$$\frac{1}{D^4+2D^2+1} x^3 = \left[1+(D^4+2D^2)\right]^{-1} x^3$$

$$= \left[1 + D^4 + 2D^2\right] x^3$$

$$= x^3 + 2D^2(x^3) = x^3 + 2[6x] = x^3 + 12x \quad \text{— (3)}$$

$$\frac{1}{D^4+2D^2+1} x^2\cos^2 x = \frac{1}{D^4+2D^2+1} x^2\left[\frac{1+\cos 2x}{2}\right]$$

$$= \frac{1}{2}\frac{1}{D^4+2D^2+1} x^2 + \frac{1}{2}\frac{1}{D^4+2D^2+1} x^2\cos 2x$$

$$\frac{1}{2}\frac{1}{D^4+2D^2+1} x^2 = \frac{1}{2}\left[\frac{1}{1+(D^4+2D^2)}\right] x^2 = \frac{1}{2}\left[1+(D^4+2D^2)\right]^{-1} x^2$$

$$\frac{1}{2}\left[1 + D^4 + 2D^2\right] x^2 = \frac{1}{2}\left[x^2 + 2D^2(x^2)\right] = \frac{1}{2}\left[x^2+2\right]$$

$$= \frac{x^2}{2} + 1$$

$$\frac{1}{2}\frac{1}{D^4+2D^2+1} x^2\cos 2x = \frac{1}{2}\frac{1}{D^4+2D^2+1} e^{0x} x^2\cos 2x$$

W.K.T, $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V.$

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$$\frac{1}{2}\frac{1}{D^4+2D^2+1}e^{0x}x^2\cos 2x$$

$$=\frac{1}{2}e^{0x}\frac{1}{D^4+2D^2+1}x^2\cos 2x$$

$$=\frac{1}{2}.e^{0x}\cancel{\frac{1}{D^4+2D^2+1}}\ R.P\ \frac{1}{D^4+2D^2+1}x^2e^{i2x}$$

$$\cancel{=\frac{1}{2}e^{0x}\frac{1}{D^4+2D^2+1}\frac{1}{16i^4+4i^2+1}x^2e^{2ix}}$$

$$\cancel{=\frac{1}{2}\frac{1}{16-4+1}x^2e^{2ix}=\frac{1}{2}\frac{1}{13}x^2e^{2ix}}$$

$$=\frac{1}{2}R.P\ .e^{2ix}\frac{1}{(D+2i)^4+2(D+2i)^2+1}x^2$$

$$=\frac{1}{2}.R.P\ e^{2ix}\left[\frac{x^2}{2}-2\right]$$

$$=R.P\left[\cos 2x+i\sin 2x\right]\left[\frac{x^2}{2}-2\right]$$

$$=\cos 2x\left(\frac{x^2}{2}\right)-2\cos 2x.$$

$$y_p=\frac{1}{4}e^{-x}+\left(\frac{x^2}{2}-2\right)+\cos 2x\left(\frac{x^2}{2}\right)-2\cos 2x+x^3+12x.$$

$\therefore$ The G.S is $y=y_c+y_p$

$$y=c_1\cos x-c_2\sin x+c_3\cos x-c_4\sin x+\frac{e^{-x}}{4}+\left(\frac{x^2}{2}-2\right)$$
$$+\cos 2x\left(\frac{x^2}{2}\right)-2\cos 2x+x^3+12x$$

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General Method :-

1) Solve $(D^2+3D+2)y = e^{e^x}$

sol:- Let $f(D) = D^2+3D+2$, $Q = e^{e^x}$

The Auxiliary equation is $f(m)=0$ i.e $m^2+3m+2=0$

$$(m+1)(m+2)=0$$

$$m = -1, -2$$

$$C.F = y_c = c_1 e^{-x} + c_2 e^{-2x}.$$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$y_p = \frac{1}{D^2+3D+2} e^{e^x} = \frac{1}{(D+2)(D+1)} e^{e^x}$$

$$= \frac{1}{D+2} \frac{1}{D+1} e^{e^x}. \qquad \left[\because \frac{1}{D+\alpha} Q = e^{-\alpha x} \int Q e^{\alpha x}\right]$$

$$= \frac{1}{D+2} \left[e^{-x} \int e^{e^x} e^x \, dx\right] \qquad \left[\because \int e^{f(x)} f'(x) dx = e^{f(x)}\right]$$

$$= \frac{1}{D+2} (e^{-x} e^{e^x})$$

$$= e^{-2x} \int e^{-x} e^{e^x} e^{2x} dx.$$

$$= e^{-2x} \int e^{e^x} e^x dx.$$

$$y_p = e^{-2x} e^{e^x}.$$

$\therefore$ The general solution of ① is $y = y_c + y_p$.

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}.$$

2) Solve $(D^2-1)y = (1+e^{-x})^{-2}$.

sol: Let $f(D) = D^2-1$ $\quad Q = (1+e^{-x})^{-2}$

The Auxiliary equation is $f(m)=0$ i.e $m^2-1=0$ $\quad m^2=1$

$$m = \pm 1$$

$$C.F = y_c = c_1 e^x + c_2 e^{-x}.$$

$$P.I = y_p = \frac{1}{f(D)} Q = \frac{1}{D^2-1} (1+e^{-x})^{-2} = \frac{1}{(D-1)(D+1)} \frac{1}{(1+e^{-x})^2}$$

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$$= \frac{1}{2}\left(\frac{1}{D-1} - \frac{1}{D+1}\right)\frac{1}{(1+e^{-x})^2}.$$

$$\frac{1}{D+1}\frac{1}{(1+e^{-x})^2} = e^{-x}\int \frac{1}{(1+e^{-x})^2} e^{x}\,dx \qquad \left[\because \frac{1}{D+\alpha}\cdot Q = e^{-\alpha x}\int Q e^{\alpha x}\right.$$

$$= e^{-x}\int \frac{e^{2x}}{(e^{x}+1)^2} e^{x}\,dx$$

Let $1+e^{x} = t \quad e^{x}dx = dt.$

$$= e^{-x}\int \frac{(t-1)^2}{t^2}\,dt$$

$$= e^{-x}\int\left(1-\frac{2}{t}+\frac{1}{t^2}\right)dt$$

$$= e^{-x}\left(t - 2\log t - \frac{1}{t}\right)$$

$$= e^{-x}\left[1+\cancel{e^{x}} - 2\log|1+e^{x}| - \frac{1}{1+e^{x}}\right]$$

$$= e^{-x} + 1 - 2e^{-x}\log|1+e^{x}| - \frac{e^{-x}}{1+e^{x}}$$

$$\frac{1}{D-1}\cdot\frac{1}{(1+e^{-x})^2} = e^{x}\int\frac{1}{(1+e^{-x})^2}e^{-x}\,dx \qquad \left[\because \frac{1}{D-\alpha}Q = e^{\alpha x}\int Q e^{-\alpha x}dx\right]$$

Let $1+e^{-x} = t \quad -e^{-x}dx = dt \implies e^{-x}dx = -dt.$

$$\frac{1}{(D-1)}\frac{1}{(1+e^{-x})^2} = e^{x}\int\frac{1}{t^2}(-dt) = e^{x}.\frac{1}{t}.$$

$$= \frac{e^{x}}{1+e^{-x}}$$

$$\therefore P.I = y_p = \frac{1}{2}\left[\frac{e^{x}}{1+e^{-x}} - e^{-x} - 1 + 2e^{-x}\log|1+e^{x}| + \frac{e^{-x}}{1+e^{x}}\right]$$

Hence the general solution is $y = y_c + y_p$.

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2}\left[\frac{e^{x}}{1+e^{-x}} - e^{-x} - 1 + 2e^{-x}\log|1+e^{x}| + \frac{e^{-x}}{1+e^{x}}\right].$$

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Solve $(D^2-3D+2)y = \sin(e^{-x})$.

Sol:- Given that $(D^2-3D+2)y = \sin(e^{-x})$.

The given differential equation is of the form $f(D)y = Q$

$$f(D) = D^2-3D+2, \quad Q = \sin(e^{-x})$$

The Auxiliary equation is $f(m)=0$ i.e $m^2-3m+2=0$

$$(m-1)(m-2)=0$$

$$m = 1, 2$$

The roots are real and distinct.

$$\therefore \quad C.F = y_c = c_1 e^{x} + c_2 e^{2x}.$$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$y_p = \frac{1}{D^2-3D+2}\sin(e^{-x}) = \frac{1}{(D-1)(D-2)}\sin(e^{-x}) = \left[\frac{1}{D-2} - \frac{1}{D-1}\right]\sin(e^{-x})$$

$$= \frac{1}{D-2}\sin e^{-x} - \frac{1}{D-1}\sin e^{-x}. \quad \text{——} ①$$

$$\frac{1}{D-2}\sin(e^{-x}) = e^{2x}\int e^{-2x}\sin(e^{-x})\,dx$$

Put $e^{-x} = t$
$e^{-x}dx = -dt$.

$$= e^{2x}\int -t\sin t\,dt$$

$$= -e^{2x}\int t\sin t\,dt.$$

$$= -e^{2x}\left[t(-\cos t) - \int (-\cos t)\,dt\right]$$

$$= -e^{2x}\left[-t\cos t + \sin t\right] \quad \text{——} ②$$

$$\frac{1}{D-2}\sin(e^{-x}) = -e^{2x}\left[-e^{-x}\cos(e^{-x}) + \sin(e^{-x})\right] = e^{x}\cos(e^{-x}) - e^{2x}\sin(e^{-x}).$$

$$\frac{1}{D-1}\sin(e^{-x}) = e^{x}\int e^{-x}\sin(e^{-x})\,dx$$

Put $e^{-x} = t$
$e^{-x}dx = -dt$

$$= e^{x}\int -\sin t\,dt = -e^{x}[-\cos t]$$

$$\frac{1}{D-1}\sin(e^{-x}) = e^{x}\cos(e^{-x}) \quad \text{——} ③$$

Sub ② and ③ in ①, we get $y_p = \left[e^{x}\cos(e^{-x}) - e^{2x}\sin(e^{-x})\right] - e^{x}(\cos e^{-x})$

$$y_p = -e^{2x}\sin(e^{-x})$$

The general solution is $y = y_c + y_p$. $y = c_1 e^{x} + c_2 e^{2x} - e^{2x}\sin(e^{-x})$.

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1. Solve $(D^2+D)y = \frac{1}{1+e^x}$

Sol:- Given that $(D^2+D)y = \frac{1}{1+e^x}$

The given differential equation is of the form $f(D)y = Q$

$$f(D) = D^2+D, \quad Q = \frac{1}{1+e^x}$$

The A.E is $f(m)=0$ i.e $m^2+m=0$
$m = 0, -1$

The roots are real and distinct.

$$C.F = y_c = c_1 + c_2 e^{-x}.$$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$\therefore y_p = \frac{1}{D^2+D} \frac{1}{1+e^x} = \frac{1}{D(D+1)} \frac{1}{1+e^x}$$

$$= \left[\frac{1}{D} - \frac{1}{D+1}\right] \frac{1}{1+e^x}$$

$$= \frac{1}{D} \frac{1}{1+e^x} - \frac{1}{D+1} \frac{1}{1+e^x}$$

$$= \int \frac{dx}{1+e^x} - e^{-x} \int e^x \cdot \frac{1}{1+e^x} dx.$$

$$= -\int \frac{-e^{-x}}{e^{-x}+1} dx - e^{-x} \int \frac{e^x dx}{1+e^x}$$

$$y_p = -\log|e^{-x}+1| - e^{-x} \log|1+e^x|.$$

$\therefore$ The general solution of (1) is $y = y_c + y_p$

$$y = c_1 + c_2 e^{-x} - \log|e^{-x}+1| - e^{-x} \log|1+e^x|.$$

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Solve $(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x)$.

Sol: Let $f(D) = D^2+5D+6$ $\quad Q(x) = e^{-2x}\sec^2 x(1+2\tan x)$.

The Auxiliary equation is $f(m)=0$ i.e $m^2+5m+6=0$

$(m+2)(m+3)=0$

$m=-2,-3$.

$$C.F = y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

$$P.I = y_p = \frac{1}{f(D)} Q$$

$$y_p = \frac{1}{D^2+5D+6} e^{-2x}\sec^2 x(1+2\tan x)$$

$$= \frac{1}{(D+2)(D+3)} e^{-2x}\sec^2 x(1+2\tan x)$$

$$= \left(\frac{1}{D+2} - \frac{1}{D+3}\right) e^{-2x}\sec^2 x(1+2\tan x)$$

$$\frac{1}{D+2} e^{-2x}\sec^2 x(1+2\tan x) = e^{-2x}\int e^{-2x}\sec^2 x(1+2\tan x)\, e^{2x}\,dx.$$

$$= e^{-2x}\int (1+2\tan x)\sec^2 x\,dx.$$

$$\because \int f(x) f'(x)\,dx = \frac{[f(x)]^2}{2}.$$

$$= \frac{e^{-2x}}{2}\cdot\frac{(1+2\tan x)^2}{2}.$$

$$\frac{1}{D+3} e^{-2x}\sec^2 x(1+2\tan x) = e^{-3x}\int e^{-2x}\sec^2 x(1+2\tan x)\, e^{3x}\,dx.$$

$$= e^{-3x}\left[\int e^{x}\sec^2 x\,dx + \int e^{x}\sec^2 x.\,2\tan x\,dx\right].$$

$$= e^{-3x}\left[e^{x}\sec^2 x - \int e^{x}\, 2\sec x.\cancel{\sec x\tan x\,dx} + \int e^{x}\cancel{\sec^2 x}.\,2\tan x\,dx\right]$$

$$= e^{-2x}\sec^2 x.$$

$$\therefore P.I = \frac{e^{-2x}}{4}(1+2\tan x)^2 - e^{-2x}\sec^2 x.$$

$$= \frac{e^{-2x}}{4}(1+4\tan^2 x+4\tan x) - e^{-2x}(1+\tan^2 x) = \frac{e^{-2x}}{4}(4\tan x-3)$$

$$= \frac{e^{-2x}}{4}(1+$$

$\therefore$ The general solution is $y = y_c + y_p$.

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^{-2x}}{4}(4\tan x - 3).$$

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Method of Variation of Parameters :-

An equation of the form $\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = Q(x)$. Where P_1, P_2 and Q are real valued functions of x. is called the linear differential of 2nd order with variable coefficients.

Working Procedure :-

Step 1 :- Reduce the given D.E to the standard form

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = Q(x)$$

Step 2 :- Find the general solution of $\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$. and let the solution be $y_c = c_1 u(x) + c_2 v(x)$.

Step 3 :- Take particular integral $P.I = y_p = A u(x) + B v(x)$

Where A and B are functions of x.

Step 4 :- Find $W = uv' - vu'$ and observe that $W(u,v) \neq 0$. Which is called Wronskian.

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

Step 5 : Find A and B using $A = -\int \frac{vQ}{uv' - vu'}\,dx$ and $B = \int \frac{uQ}{uv' - vu'}\,dx$

The general solution of a given D.E is $y = y_c + y_p$.

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→ Solve $(D^2+a^2)y = \tan ax$ by the method of variation of parameters.

Sol:- Given that $(D^2+a^2)y = \tan ax$.

Which is of the form $f(D)y = Q$

$f(D) = D^2+a^2 \quad Q = \tan ax$.

An Auxiliary Eqn. is $f(m)=0$ ie $m^2+a^2=0$ The roots are imaginary

$m = \pm ai$

$$C.F = y_c = c_1 \cos ax + c_2 \sin ax.$$

Which is of the form $y_c = c_1 u(x) + c_2 v(x)$

$u = \cos ax \quad \& \quad v = \sin ax$.

$u' = -a\sin ax \quad v' = a\cos ax$

$uv' - vu' = a\cos^2 ax + a\sin^2 ax = a \neq 0$.

Let $P.I = y_p = A u(x) + B v(x)$

$y_p = A\cos ax + B\sin ax$.

Where $A = -\int \frac{vQ}{uv'-vu'}dx \quad B = \int \frac{uQ}{uv'-vu'}dx$

$$A = -\int \frac{\sin ax \tan ax}{a}dx = -\int \frac{\sin^2 ax}{a\cos ax}dx$$

$$= -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}dx = -\frac{1}{a}\left[\int \sec ax\,dx - \int \cos ax\,dx\right]$$

$$A = -\frac{1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax.$$

$$B = \int \frac{uQ}{uv'-vu'}dx = \int \frac{\cos ax \tan ax}{a}dx = \frac{1}{a}\int \sin ax\,dx = -\frac{1}{a^2}\cos ax.$$

Sub. A and B in y_p, we get

$$y_p = \left[-\frac{1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax\right]\cos ax - \frac{1}{a^2}\cos ax \sin ax.$$

∴ The general solution is given by $y = y_c + y_p$

$$y = c_1\cos ax + c_2\sin ax + \left[-\frac{1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax\right]\cos ax - \frac{1}{a^2}\cos ax \sin ax.$$

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→ Solve $(D^2-1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$.

Sol: Let $f(D) = D^2-1$, $Q = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$.

The Auxiliary equation is $f(m)=0$. i.e $m^2-1=0 \Rightarrow m^2=1$

$m = \pm 1$.

$$C.F = y_c = c_1 e^{x} + c_2 e^{-x}.$$

Let particular solution is $P.I = y_p = A(x)\,e^{x} + B(x)\,e^{-x}$

Let $u(x) = e^{x}$ $\quad v(x) = e^{-x}$

$u'(x) = e^{x}$ $\quad v'(x) = -e^{-x}$

Wronskian $W = uv' - vu' = -e^{0} - e^{0} = -2 \neq 0$

$$A = -\int \frac{vQ}{W}\,dx$$

$$= -\int \frac{e^{-x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]}{-2}\,dx.$$

Put $e^{-x} = t \Rightarrow -e^{-x}dx = dt$.

$$A = \int \frac{-[t\sin t + \cos t]}{2}\,dt$$

$$= -\frac{1}{2}\int (t\sin t + \cos t)\,dt$$

$$= -\frac{1}{2}\left[t(-\cos t) - (-\sin t) + \sin t\right]$$

$$= \frac{1}{2}\,t\cos t - \sin t$$

$$= \frac{1}{2}\,e^{-x}\cos(e^{-x}) - \sin(e^{-x}).$$

$$B = \int \frac{uQ}{W}\,dx.$$

$$= \int \frac{e^{x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]}{-2}\,dx.$$

$$= \int \frac{e^{x}\left[\cos e^{-x} + e^{-x}\sin(e^{-x})\right]}{-2}\,dx \quad \because \int e^{x}\{f(x)+f'(x)\}dx = e^{x}f(x).$$

here $f(x) = \cos e^{-x}$.

$$= -\frac{1}{2}\,e^{x}\cos(e^{-x})$$

$$P.I = -e^{-x}\sin(e^{-x}).$$

∴ The general solution is $y = y_c + y_p$ $\quad y = c_1 e^{x} + c_2 e^{-x} - e^{-x}\sin e^{-x}$.

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→ Solve $(D^2+3D+2)y = e^{e^x}$. by Method of variation of parameters.

Sol:- Given that $(D^2+3D+2)y = e^{e^x}$.

$f(D) = D^2+3D+2 \qquad Q = e^{e^x}$

The A.E is $f(m)=0$ i.e $m^2+3m+2=0$.

$m = -1, -2$.

The roots are real and distinct.

$C.F = y_c = c_1 e^{-x} + c_2 e^{-2x}$.

$u(x) = e^{-x} \qquad v(x) = e^{-2x}$

$u'(x) = -e^{-x} \qquad v'(x) = -2e^{-2x}$

Wronskian $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv'-vu' = -2e^{-3x}+e^{-3x} = -e^{-3x} \neq 0$.

Let $P.I = y_p = A\,u(x) + B\,v(x)$ —

$y_p = A e^{-x} + B e^{-2x}$ ——— (1)

$A = -\int \frac{vQ\,dx}{uv'-vu'}$

$A = -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}} dx$

$= \int e^x e^{e^x} dx$. $\qquad e^x = t$
$\qquad e^x dx = dt$

$= \int e^t dt$

$A = e^t = e^{e^x}$

$B = \int \frac{uQ\,dx}{uv'-vu'}$

$= \int \frac{e^{-x} e^{e^x}}{-e^{-3x}} dx = -\int e^{2x} e^{e^x} dx \qquad e^x = t$
$\qquad e^x dx = dt$

$= -\int e^x e^x e^{e^x} dx$

$= -\int t e^t dt$.

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$$= -(te^{t} - e^{t})$$

$$B = -e^{x}e^{e^{x}} + e^{e^{x}}$$

Sub. A and B in ①, we get

$$P.I = y_p = e^{e^{x}}e^{-x} + e^{-2x}(-e^{x}e^{e^{x}} + e^{e^{x}})$$

$$y_p = e^{-2x}e^{e^{x}}$$

$\therefore$ The G. sol. is $y = y_c + y_p$.

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}e^{e^{x}}.$$

→ Solve $(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x)$ by Method of variation of parameters.

Sol:- Given that $(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x)$.

$f(D) = D^2+5D+6$, $Q = e^{-2x}\sec^2 x(1+2\tan x)$.

The A.E is $f(m)=0$ i.e $m^2+5m+6=0$

$$m = -2, -3$$

The roots are real and distinct.

$$C.F = y_c = c_1 e^{-2x} + c_2 e^{-3x}.$$

$$u(x) = e^{-2x} \qquad v(x) = e^{-3x}$$

$$u'(x) = -2e^{-2x} \qquad v'(x) = -3e^{-3x}.$$

Wronskian $W = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = uv' - vu'$

$$= -3e^{-5x} + 2e^{-5x} = -e^{-5x} \neq 0.$$

Let $P.I = y_p = Au(x) + Bv(x)$

$$y_p = Ae^{-2x} + Be^{-3x}.$$

$$A = -\int \frac{vQ\,dx}{uv' - vu'}$$

$$= -\int \frac{e^{-3x}\cdot e^{-2x}\sec^2 x(1+2\tan x)}{-e^{-5x}}\,dx.$$

$$= \int (1+2\tan x)\sec^2 x\,dx.$$

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$$= \frac{1}{2}\int (1+2\tan x)\, 2\sec^2 x\, dx$$

$$= \frac{1}{2}\frac{(1+2\tan x)^2}{2}$$

$$\left[\int f(x) f'(x)\, dx = \frac{\{f(x)\}^2}{2} \qquad f(x) = 1+2\tan x\right.$$

$$u = \frac{1}{4}(1+2\tan x)^2$$

$$v = \int \frac{uQ\, dx}{uv' - vu'}$$

$$= \int \frac{e^{-2x} \cdot e^{-2x} \sec^2 x (1+2\tan x)}{-e^{-5x}}\, dx$$

$$= -\int e^{x} \sec^2 x (1+2\tan x)\, dx$$

$$= -\int e^{x} (\sec^2 x + 2\sec^2 x \tan x)\, dx.$$

$$= -e^{x}\sec^2 x$$

$$\left[\because \int e^{x}[f(x)+f'(x)]\, dx = e^{x} f(x) \qquad f(x) = \sec^2 x\right.$$

Sub. A and B in ①, we get

$$P.I = y_p = e^{-2x} \cdot \frac{1}{4}(1+2\tan x)^2 + e^{-3x}(-e^{x}\sec^2 x).$$

$\therefore$ The G.sol. is $y = y_c + y_p$.

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^{-2x}}{4}(1+2\tan x)^2 + e^{-2x}\sec^2 x.$$

→ Solve $(D^2+1)y = \frac{1}{1+\sin x}$

Sol:- The given D.E is $(D^2+1)y = \frac{1}{1+\sin x}$

$$f(D) = D^2+1 \qquad Q = \frac{1}{1+\sin x}.$$

The A.E is $f(m) = 0$ i.e $m^2+1 = 0$

$$m = \pm i$$

The roots are imaginary.

$$C.F = y_c = c_1 \cos x + c_2 \sin x.$$

$$u(x) = \cos x \qquad v(x) = \sin x$$

$$u'(x) = -\sin x \qquad v'(x) = \cos x.$$

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Wronskian $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu' = \cos^2 x + \sin^2 x = 1 \neq 0$.

Let $P.I = y_p = A\,u(x) + B\,v(x)$

$$y_p = A\cos x + B\sin x. \quad \text{(1)}$$

$$A = -\int \frac{vQ\,dx}{uv' - vu'}$$

$$A = -\int \sin x \cdot \frac{1}{1+\sin x}\,dx.$$

$$= -\int \frac{\sin x}{1+\sin x} \cdot \frac{1-\sin x}{1-\sin x}\,dx. = -\int \frac{\sin x - \sin^2 x}{1-\sin^2 x}\,dx.$$

$$= -\int \frac{\sin x - \sin^2 x}{\cos^2 x}\,dx$$

$$= -\int (\tan x \sec x - \tan^2 x)\,dx.$$

$$= -\int (\tan x \sec x - \sec^2 x + 1)\,dx.$$

$$A = -(\sec x - \tan x + x)$$

$$B = \int \frac{uQ\,dx}{uv' - vu'}$$

$$= \int \cos x \cdot \frac{1}{1+\sin x}\,dx$$

$$B = \log|1+\sin x|$$

Sub. A and B in (1), we get

$$P.I = y_p = -(\sec x - \tan x + x)\cos x + \sin x \log|1+\sin x|.$$

$\therefore$ The G.sol. is $y = y_c + y_p$

$$y = c_1\cos x + c_2\sin x - \cos x(\sec x - \tan x + x) + \sin x \log|1+\sin x|.$$

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→ Solve $(D^2-2D+2)y = e^x \tan x$ by the Method of Variation of parameters.

Sol: Given that $(D^2-2D+2)y = e^x \tan x$.

$f(D) = D^2-2D+2 \qquad Q = e^x \tan x$

The A.E is $f(m)=0$ i.e $m^2-2m+2=0$

$$m = 1 \pm i$$

The roots are Imaginary.

$$C.F = y_c = e^x (c_1 \cos x + c_2 \sin x)$$

$u(x) = e^x \cos x \qquad v(x) = e^x \sin x$.

$u'(x) = e^x \cos x - e^x \sin x \qquad v'(x) = e^x \sin x + e^x \cos x$.

Wronskian $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$

$$= e^x \cos x (e^x \sin x + e^x \cos x) - e^x \sin x (e^x \cos x - e^x \sin x)$$

$$= e^{2x} \neq 0.$$

Let $P.I = y_p = A\,u(x) + B\,v(x)$

$$y_p = A\, e^x \cos x + B\, e^x \sin x.$$

$$A = -\int \frac{vQ\,dx}{uv'-vu'}$$

$$= -\int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}} dx. = -\int \sin x \cdot \frac{\sin x}{\cos x} dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx = -\int \frac{1-\cos^2 x}{\cos x} dx.$$

$$= -\int \sec x\,dx + \int \cos x\,dx$$

$$= -\log|\sec x + \tan x| + \sin x.$$

$$B = \int \frac{uQ\,dx}{uv'-vu'}$$

$$= \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} dx. = \int \sin x\,dx$$

$$B = -\cos x.$$

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Sub. A and B in (1), we get

$$P.I = y_p = \left[-\log|\sec x + \tan x| + \sin x\right] e^x \cos x + (-\cos x)\, e^x \sin x.$$

$$y_p = -e^x \cos x \log|\sec x + \tan x|.$$

The G.sol. is $y = y_c + y_p$

$$y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log|\sec x + \tan x|.$$

→ Solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ by the Method of variation of parameters.

Sol: An operator form of the given D.E is $(D^2-1)y = \frac{2}{1+e^x}$

$f(D) = D^2 - 1$ $\quad Q = \frac{2}{1+e^x}$

The A.E is $f(m) = 0$ i.e $m^2 - 1 = 0$

$$m = \pm 1$$

The roots are real and distinct

$$C.F = y_c = c_1 e^x + c_2 e^{-x}.$$

$u(x) = e^x \quad v(x) = e^{-x}$

$u'(x) = e^x \quad v'(x) = -e^{-x}$

Wronskian $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu' = e^x(-e^{-x}) - e^x e^{-x} = -2 \neq 0$

let $P.I = y_p = A\, u(x) + B\, v(x)$.

$$y_p = A e^x + B e^{-x}$$

$$A = -\int \frac{vQ\, dx}{uv' - vu'}$$

$$= -\int \frac{e^{-x} \cdot \frac{2}{1+e^x}}{-2}\, dx \;.\; = \int \frac{1}{e^x(1+e^x)}\, dx$$

$$= +\int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx$$

$$= \int e^{-x}\, dx + \int \frac{-e^{-x}}{e^{-x}+1}\, dx$$

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$$= \frac{e^{-x}}{-1} + \log|1+e^{-x}|$$

$$= \log|1+e^{-x}| - e^{-x}.$$

$$B = \int \frac{uQ\,dx}{uv' - vu'}$$

$$= \int \frac{e^{x}\frac{2}{1+e^{x}}}{-2}\,dx$$

$$= -\int \frac{e^{x}}{1+e^{x}}\,dx$$

$$B = -\log|1+e^{x}|.$$

Sub. A and B in ①, we get

$$P.I = y_p = e^{x}\log|1+e^{-x}| - 1 + e^{-x}\log|1+e^{x}|$$

The G. sol. is $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{-x} + e^{x}\log|1+e^{-x}| - 1 - e^{-x}\log|1+e^{x}|.$$

Solve $y'' - 2y' + y = e^{x}\log x$.

Given that $y'' - 2y' + y = e^{x}\log x$.

An operator form of given eqn. is $(D^2 - 2D + 1)y = e^{x}\log x$.

Which is of the form $f(D)y = Q$.

Where $f(D) = D^2 - 2D + 1$ $\quad Q = e^{x}\log x$.

An auxiliary equation is $f(m) = 0$ i.e $m^2 - 2m + 1 = 0$

$$m = 1, 1$$

The roots are real and repeated

$$C.F = y_c = (c_1 x^0 + c_2 x)e^{x}.$$

Which is of the form $y_c = c_1 u(x) + c_2 v(x)$

$$u(x) = e^{x} \qquad v(x) = xe^{x}.$$

$$u' = e^{x} \qquad v' = e^{x} + xe^{x}.$$

$$W = uv' - vu' = e^{x}(e^{x} + xe^{x}) - e^{x}.xe^{x} = e^{2x} \neq 0.$$

Let P.I be, $P.I = y_p = Au(x) + Bv(x)$

$$y_p = Ae^x + Bxe^x.$$

Where $A = -\int \frac{vQ}{uv' - vu'} dx$ $\quad B = \int \frac{uQ}{uv' - vu'} dx.$

$$A = -\int \frac{xe^x \, e^x \log x}{e^{2x}} \cdot dx = -\int x \log x \, dx.$$

$$A = -\left[(\log x)\int x dx - \int\left[\frac{1}{x} \cdot \int x dx\right] dx\right]$$

$$A = -\left[\log x \cdot \frac{x^2}{2} - \int \frac{x}{2} dx\right]$$

$$A = -\frac{x^2}{2}\log x + \frac{x^2}{4}$$

$$B = \int \frac{uQ}{uv' - vu'} dx$$

$$B = \int \frac{e^x e^x \log x}{e^{2x}} = \int \log x \, dx$$

$$B = x\log x - x$$

Sub. A and B in y_p:

$$y_p = \left[\frac{x^2}{4} - \frac{x^2}{2}\log x\right]e^x + \left[x\log x - x\right]e^x . x$$

$$y_p = \left[\frac{x^2}{4} - \frac{x^2}{2}\log x\right] e^x + x^2\left[\log x - 1\right]e^x$$

The General sol. is $y = y_c + y_p$.

$$y = (c_1 x^0 + c_2 x)e^x + \left(\frac{x^2}{4} - \frac{x^2}{2}\log x\right)e^x + x^2(\log x - 1)e^x.$$

→ Solve $(D^2 - 3D + 2)y = \cos(e^{-x})$

Sol:- Given that $(D^2 - 3D + 2)y = \cos(e^{-x})$.

Which is of the form, $f(D)y = Q$.

Where $f(D) = D^2 - 3D + 2$ $\quad Q = \cos(e^{-x})$

An Auxiliary Equation is $f(m) = 0$ i.e $m^2 - 3m + 2 = 0$

$$m = 1, 2.$$

The roots are real and distinct

$$y_c = c_1 e^{2x} + c_2 e^x$$

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Which is of the form $y_c = c_1 u(x) + c_2 v(x)$

$$u = e^{-2x} \quad v = e^{-x}.$$

$$u' = -2e^{-2x} \cdot \quad v' = -e^{-x}$$

$$W = uv' - vu' = e^{-2x}\cdot e^{-x} - 2e^{-2x}e^{-x} = -e^{-3x} \neq 0$$

Let $P.I = y_p = A\,u(x) + B\,v(x)$

$$y_p = A\,e^{-2x} + B\,e^{-x}.$$

Where $A = -\int \frac{vQ}{uv' - vu'}\,dx \qquad B = \int \frac{uQ}{uv' - vu'}\,dx$

$$A = -\int \frac{e^{-x}\cos(e^{-x})}{-e^{-3x}}\,dx = \int (e^{-x})^2 \cos(e^{-x})\,dx$$

Put $e^{-x} = t$

$-e^{-x}\,dx = dt$

$e^{-x}\,dx = -dt$

$$A = -\int \overset{f}{t}\,\overset{g}{\cos t}\,dt$$

$$A = -\left[t(\sin t) - 1(-\cos t)\right]$$

$$A = -(t\sin t + \cos t)$$

$$A = -\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]$$

$$B = \int \frac{uQ}{uv' - vu'}\,dx$$

$$= \int \frac{e^{-2x}\cdot\cos(e^{-x})}{-e^{-3x}}\,dx = -\int e^{x}\cos(e^{-x})\,dx.$$

Put $e^{-x} = t$

$-e^{-x}\,dx = dt$

$$= \int \cos t\,dt = \sin t$$

$$B = \sin(e^{-x})$$

Sub. A and B in y_p, we get

$$y_p = -e^{-2x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right] + e^{-x}\sin(e^{-x})$$

$$y_p = -e^{-2x}\cos(e^{-x})$$

$\therefore$ The general sol. is $y = y_c + y_p$

$$y = c_1 e^{-2x} + c_2 e^{-x} - e^{-2x}\cos(e^{-x}).$$

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MODULE -V

PARTIAL DIFFERENTIAL EQUATIONS

PARTIAL DIFFERENTIAL EQUATIONS

Partial Differential Equation :—

The differential equation having differential co efficients w.r. to two or more independent variables is said to be partial differential equation.

Therefore a partial differential equation contains one dependent and two or more independent variables.

In a relation $z = f(x,y)$. x, y are independent variables and z is a. dependent variable so that we have to find the partial derivative of z w.r.t x, y.

Eg :- $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Order and Degree of the partial differential equation :—

The order of a partial differential equation is the order of the highest partial derivative occuring in the equation and its degree is the power of the highest order partial derivative in the equation

Eg:- $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$ is of first order and first degree.

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 0$ is of second order and first degree.

For a relation $z = f(x, y)$.

The partial derivative of z, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ will be denoted by p and q respectively.

The Second partial derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ are denoted by r, t and s respectively.

So that $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$.

Formation of Partial Differential Equations :—

If the number of arbitrary constants to be eliminated is equal to the number of independent variables involved in the equation, then we obtain a partial differential equation of first order.

11) Find the differential equation of all spheres of radius 5 having their centres in the xy-plane.

Sol:- The equation of the family of spheres having centre at (a,b,c) and radius r are. $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.

The equation of the family of spheres having their centres in the xy-plane and having radius 5 is

$$(x-a)^2 + (y-b)^2 + z^2 = 25 \quad \text{—— (1)}$$

Diff (1) partially w.r.t x and y, we get

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

$$(x-a) = -zp \quad \text{—— (2)}$$

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$(y-b) = -zq \quad \text{—— (3)}$$

Sub. the values of $(x-a)$ and $(y-b)$ in (1), we get.

$$z^2p^2 + z^2q^2 + z^2 = 25$$

$$z^2(p^2+q^2+1) = 25$$

This is the required diff. equation.

Form the partial differential equation by eliminating the arbitrary constants a, b from $z = a \log\left[\frac{b(y-1)}{1-x}\right]$

Sol:- Given that $z = a\log\left[\frac{b(y-1)}{(1-x)}\right]$

$$z = a\left[\log b(y-1) - \log(1-x)\right]$$

$$z = a\log b + a\log(y-1) - a\log(1-x) \quad \text{--- (1)}$$

The number of arbitrary constants to be eliminated is equal to the number of independent variables involved in the equation then we obtain a partial differential equation of first order.

Diff (1) w.r.t x and y partially, we get

$$\frac{\partial z}{\partial x} = -\frac{a}{1-x}(-1)$$

$$p = \frac{a}{1-x}.$$

$$p(1-x) = a \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = \frac{a}{y-1}$$

$$q = \frac{a}{y-1}$$

$$q(y-1) = a \quad \text{--- (3)}$$

From (2) and (3), we get

$$p(1-x) = q(y-1)$$

$$px + qy = p + q.$$

This is the required partial differential equation.

Form the diff. equation of all planes having equal intercepts on x and y axes.

Sol:- We know that the equation of the plane having intercepts on x, y and z axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

The equation of the plane having equal intercepts on x and y axes is $\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1$ —— ①

The number of arbitrary constants to be eliminated is equal the number of independent variables involved in the equation, so we obtain a partial differential equation of first order

Diff ① w.r.t x and y partially, we get

$$\frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial x} = 0$$

$$\frac{1}{a} = -\frac{p}{c} \text{ —— ②}$$

$$\frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial y} = 0.$$

$$\frac{1}{a} = -\frac{q}{c} \text{ —— ③}$$

From ② and ③, we get

$$-\frac{p}{c} = \frac{-q}{c}$$

$$p = q.$$

This is the required partial diff. equation.

Note:- If the number of arbitrary constants to be eliminated is more than the number of independent variables involved in the equation then the partial differential equation is second or higher order.

Eg:- (1) Form the partial differential equation of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol:- Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— ①

Where a, b, c are arbitrary constants.

Here we have to treat z as a fun. of x and y

The No. of arbitrary constants are three is more than the number of independent variables are two. So we obtain a partial differential eqn of second or higher order.

Diff ① partially w.r.t 'x', we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$\frac{x}{a^2}+\frac{z}{c^2}p=0 \quad \text{—— ②}$$

Diff ① partially w.r.t 'y', we get

$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

$$\frac{y}{b^2}+\frac{z}{c^2}q=0 \quad \text{—— ③}$$

Diff ② partially w.r.t 'y', we get

$$\frac{1}{c^2}\left[z\frac{\partial p}{\partial y}+p\frac{\partial z}{\partial y}\right]=0$$

$$z\frac{\partial p}{\partial y}+p\frac{\partial z}{\partial y}=0.$$

$$z \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) + p \frac{\partial z}{\partial x} = 0$$

$$z \frac{\partial^2 z}{\partial y \partial x} + p \frac{\partial z}{\partial x} = 0$$

$$zs + pq = 0$$

Which is the required partial diff. equation of second order.

Form the partial differential equation of $ax+by+cz=0$.

Given that $ax+by+cz=0$ ——— (1)

where a, b, c are arbitrary constants.

Diff (1) partially w.r.t x and y, we get

$$a + cp = 0 \Rightarrow a = -cp \quad \text{——— (2)}$$

$$b + cq = 0 \Rightarrow b = -cq \quad \text{——— (3)}$$

Sub. (2) and (3) in (1), we get

$$-cpx + cqy + cz = 0$$

$$z = px + qy$$

Which is the required partial differential equation.

Form a partial differential equation by eliminating the arbitrary constants a, b, c from the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— ①

Diff ① w.r.t 'x', partially, we get.

$$\frac{2x}{a^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial x}=0 \Rightarrow \frac{c^2}{a^2}x=-z\frac{\partial z}{\partial x} \quad \text{—— ②}$$

Diff ② w.r.t 'x', partially, we get.

$$\frac{c^2}{a^2}=-\left[z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2\right] \quad \text{—— ③}$$

From ②, $\frac{c^2}{a^2}=-\frac{z}{x}\frac{\partial z}{\partial x}$ —— ④.

From ③ and ④, we get.

$$-\frac{z}{x}\frac{\partial z}{\partial x}=-\left[z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2\right]$$

$$\frac{z}{x}\frac{\partial z}{\partial x}=z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2. \quad \text{—— ⑤}$$

This is required partial differential equation.

Note:- Diff. the given equation partially w.r.t 'y' twice and eliminating $\frac{c^2}{b^2}$, we obtain the equation. $\frac{z}{y}\frac{\partial z}{\partial y}=z\frac{\partial^2 z}{\partial y^2}+\left(\frac{\partial z}{\partial y}\right)^2$ —⑥ as another required partial differential equation.

Remark:—
Partial differential equations obtained by elimination of arbitrary constants are not unique.

→ Find a partial differential equation representing all planes that are at a constant perpendicular distance p from the origin.

Sol.- The equation of a plane that is at a perpendicular distance p from the origin is given by. $lx+my+nz=p$ —— ①.

Where (l,m,n) are the direction cosines of the normal to the plane which satisfy the identity $l^2+m^2+n^2=1$ —— ②.

As (l,m,n) take different values, subject to the identity ②, equation ① represents different planes all of whose perpendicular distance from the origin is p. In otherwords. ① is the cartesian equation of the given set (family) of planes.

Diff ① w.r.t x and y, we get.

$$l + n\frac{\partial z}{\partial x} = 0 \quad , \quad m + n\frac{\partial z}{\partial y} = 0.$$

$$l = -n\frac{\partial z}{\partial x} \qquad m = -n\frac{\partial z}{\partial y} \quad \text{— ③}$$

Sub. these in ②, we get

$$n^2\left(\frac{\partial z}{\partial x}\right)^2 + n^2\left(\frac{\partial z}{\partial y}\right)^2 + n^2 = 1$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{1}{n^2} \quad \text{— ④}$$

Now substituting for l and m from ③ in ①, we get

$$\left(-n\frac{\partial z}{\partial x}\right)x + \left(-n\frac{\partial z}{\partial y}\right)y + nz = p.$$

$$z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{p}{n}.$$

$$\left(z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right)^2 = \frac{p^2}{n^2}$$

Sub. for $\frac{1}{n^2}$ in ④, we get.

$$\left(z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right)^2 = p^2\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

This is the partial differential equation that represents the given set of planes.

Note :- (i) In case of arbitrary functions the order of partial diff. equation is equal to the number of arbitrary functions involved and to be eliminated from the given equation.

(ii) If the given relation b/n x, y, z contains two arbitrary functions then leaving a few exceptional cases the partial differential eqn's of higher order than the second will be formed.

(1) Form the partial differential equation of the following by eliminating arbitrary functions (a) $z = f(x^2 - y^2)$ (b) $z = f(x + at) + g(x - at)$

Sol:- (a) Given that $z = f(x^2 - y^2)$ —— ①

Where f is an arbitrary function.

Here the given equation has one arbitrary function so we obtain a first order partial differential equation.

Diff ① partially w.r.t x, we have

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2) . 2x .$$

$$p = f'(x^2 - y^2) . 2x \text{ —— ②}$$

Diff ① partially w.r.t y, we have.

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y)$$

$$q = f'(x^2 - y^2)(-2y) \text{ —— ③}$$

$$\frac{②}{③} \Longrightarrow \frac{p}{q} = \frac{f'(x^2 - y^2) . 2x}{f'(x^2 - y^2)(-2y)}$$

$$py + qx = 0.$$

Which is the required first order partial differential equation.

1(b) Given that $z = f(x+at) + g(x-at)$ —— (1)

Where f, g are arbitrary functions

Here the equation contains two arbitrary functions so we obtain P.D.E of second order.

Diff (1) partially w.r.t 'x', we get

$$\frac{\partial z}{\partial x} = f'(x+at) + g'(x-at) \quad \text{—— (2)}$$

Again diff partially w.r.t 'x', we have.

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \quad \text{—— (3)}$$

Diff (1) partially w.r.t 't', we have.

$$\frac{\partial z}{\partial t} = f'(x+at)\cdot a + g'(x-at)(-a)$$

Again diff. partially w.r.t t, we have

$$\frac{\partial^2 z}{\partial t^2} = f''(x+at)\, a^2 + g''(x-at)\, a^2$$

$$\frac{\partial^2 z}{\partial t^2} = a^2\left[f''(x+at) + g''(x-at)\right]$$

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \quad [\because \text{from (3)}]$$

Which is the required partial diff. eqn of 2^{nd} order.

Note :- If the implicit form of the equation is $f(u,v)=0$ then its explicit form can be written as $u=f(v)$ or $v=f(u)$.

) Form the differential equation of the following by eliminating arbitrary functions (a) $f(x^2+y^2, z-xy)=0$ (b) $xyz=\phi(x+y+z)$

Sol:- (a) Given that $f(x^2+y^2, z-xy)$

which is the implicit form of the equation

Its explicit form is $x^2+y^2=f(z-xy)$ — (1)

where f is an arbitrary function and treat z as a function of x and y

Diff (1) partially w.r.t 'x', we have

$$2x=f'(z-xy)\left(\frac{\partial z}{\partial x}-y\right)$$

$$2x=f'(z-xy)(p-y) \quad \text{— (2)}$$

Diff (1) partially w.r.t 'y', we have.

$$2y=f'(z-xy)\left(\frac{\partial z}{\partial y}-x\right)$$

$$2y=f'(z-xy)(q-x) \quad \text{— (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{2x}{2y}=\frac{f'(z-xy)(p-y)}{f'(z-xy)(q-x)}$$

$$(q-x)x=(p-y)y$$

$$qx-py=x^2-y^2$$

Which is the 1st order P.D.E.

1b) Sol:- Given that $xyz = \phi(x+y+z)$ —— ①

Which is in explicit form

Where ϕ is an arbitrary function and treat z is a function of x.

Diff ① partially w.r.t x

$$y\left[x\frac{\partial z}{\partial x} + z\right] = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right).$$

$$y(px+z) = \phi'(x+y+z)(1+p) \text{ —— ②}.$$

Diff ① partially w.r.t y, we have

$$x\left[y.\frac{\partial z}{\partial y} + z\right] = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial y}\right)$$

$$x(yq+z) = \phi'(x+y+z)(1+q) \text{ —— ③}$$

$$\frac{②}{③} \Rightarrow \frac{y(px+z)}{x(z+yq)} = \frac{\phi'(x+y+z)(1+p)}{\phi'(x+y+z)(1+q)}$$

$$y(px+z)(1+q) = x(z+yq)(1+p)$$

$$x(y-z)p + y(z-x)q = z(x-y)$$

Which is the required partial diff. eqn.

12) Form a partial differential equation by eliminating the arbitrary functions $f(x)$ and $g(y)$ from $z = yf(x) + xg(y)$

Sol:- Given that $z = yf(x) + xg(y)$ —— ①.

Diff ① w.r.t x and y partially, we have

$$\frac{\partial z}{\partial x} = yf'(x) + g(y)$$

$$p = yf'(x) + g(y) \text{ —— ②}$$

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$$\frac{\partial z}{\partial y} = f(x) + x g'(y)$$

$$q = f(x) + x g'(y) \quad \text{--- (3)}$$

Since the relations (1), (2) and (3) are not sufficient to eliminate f, g, f' and g' so we find the second order partial derivatives.

$$\frac{\partial^2 z}{\partial x^2} = r = y f''(x) \quad \text{--- (4)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = s = f'(x) + g'(y) \quad \text{--- (5)}.$$

$$\frac{\partial^2 z}{\partial y^2} = t = x g''(y) \quad \text{--- (6)}$$

From (2) and (3), we have

$$\left.\begin{aligned} f'(x) &= \frac{1}{y}\left[p - g(y)\right] \\ g'(y) &= \frac{1}{x}\left[q - f(x)\right] \end{aligned}\right\} \quad \text{--- (7)}$$

sub. (7) in (5), we get

$$s = \frac{1}{y}\left[p - g(y)\right] + \frac{1}{x}\left[q - f(x)\right]$$

$$xys = x\left[p - g(y)\right] + y\left[q - f(x)\right]$$

$$xys = px + qy - \left[x g(y) + y f(x)\right]$$

$$xys = px + qy - z$$

which is the required partial diff equation.

Linear Partial Differential Equations of the first order :-

A differential equation containing first order partial derivatives p and q only is called a partial differential equation of the 1^{st} order [OR] If p and q occur only in the first degree and are not multiplied together, then it is called a linear partial differential equation of 1^{st} order.

Eg:- (i) $2p + 30q = 44z$ is a linear P.D.E.

(ii) $2p^2 + 3044q^2 = 1$ is a Non linear.

Lagrange's Linear Equation :-

A linear partial differential equation of order one, involving a dependent variable z and two independent variables x and y, of the form.

$$Pp + Qq = R.$$

Where P, Q, R are functions of x, y, z is called Lagranges linear equation.

General Solution of the Linear Equation :-

We have seen that from a relation $\phi(u,v) = 0$ — ①, a linear partial differential equation $Pp + Qq = R$ — ② is derived by eliminating the arbitrary function ϕ. Suppose that ② has been derived from ① then $\phi(u,v) = 0$ is called the general solution or a general integral of the equation ②.

Working Procedure to solve $Pp + Qq = R$:-

Step 1 :- Compare the given partial differential equation with $Pp + Qq = R$ and identity P, Q and R.

Step 2 :- Write the subsidiary equations or auxiliary equations.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Step 3 :- Find any two independent solutions of the subsidiary equations let the two solutions be $u = a$ and $v = b$. Where a and b are constants.

Step 4: Now the general solution of $Pp + Qq = R$ is given by $f(u,v) = 0$ or $u = f(v)$ or $v = f(u)$.

Method of grouping :-

From the Lagrange's subsidiary equations, if the variables are separa-ted by taking any two members.

i.e $\frac{dx}{P} = \frac{dy}{Q}$ [OR] $\frac{dy}{Q} = \frac{dz}{R}$ [OR] $\frac{dz}{R} = \frac{dx}{P}$.

Then by integrating, the solution $u(x,y) = c_1$ or $v(y,z) = c_2$ or $w(z,x) = c_3$ is obtained. Then by taking any two of these solutions we can write the complete solution of Lagrange's equation.

i.e $\phi(u,v) = 0$ or $\phi(v,w) = 0$ or $\phi(u,w) = 0$

1) Solve $px + qy = z$

Sol:- Given that $px + qy = z$ —— (1)

Compare equation (1) with $Pp + Qq = R$

$$P = x,\quad Q = y,\quad R = z$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Considering the first two members.

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating bothsides, weget

$$\int \frac{dx}{x} = \int \frac{dy}{y} + \log c$$

$$\log|x| = \log|y| + \log c$$

$$\log\left|\frac{x}{y}\right| = \log c$$

$$\frac{x}{y} = c = u(x,y)$$

Considering the last two members,

we have $\frac{dy}{y} = \frac{dz}{z}$

Integrating both sides, we get

$$\int \frac{dy}{y} = \int \frac{dz}{z} + \log c_1$$

$$\log|y| = \log|z| + \log c_1$$

$$\log\left|\frac{y}{z}\right| = \log c_1$$

$$\frac{y}{z} = c_1 = v(y,z)$$

$\therefore$ The complete solution of given eqn is $\phi(u,v)=0$

i.e $\phi\left(\frac{x}{y}, \frac{y}{z}\right)=0$.

(2) Solve $p\tan x + q\tan y = \tan z$.

Sol:- Given that $p\tan x + q\tan y = \tan z$

The auxiliary equations are $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

Taking first two members, we have

$$\cot x\, dx = \cot y\, dy$$

Integrating bothsides, we get

$$\int \cot x\, dx = \int \cot y\, dy + \log c_1$$

$$\log|\sin x| = \log|\sin y| + \log c_1$$

$$\log\left|\frac{\sin x}{\sin y}\right| = \log c_1$$

$$\frac{\sin x}{\sin y} = c_1$$

Taking last two members, we have

$$\cot y\, dy = \cot z\, dz$$

Integrating bothsides we get

$$\int \cot y\,dy = \int \cot z\,dz + \log c_2$$

$$\log|\sin y| = \log|\sin z| + \log c_2$$

$$\log\left|\frac{\sin y}{\sin z}\right| = \log c_2$$

$$\frac{\sin y}{\sin z} = c_2$$

$\therefore$ The general solution of ① is

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0.$$

Method of Multipliers :—

Consider the Lagrange's auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

By an algebraic principle choosing a proper set of multipliers l, m, n which are not necessarily constants.

We have each ratio $= \dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$ such that $lP + mQ + nR = 0$.

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$ ($\because$ on cross multiplication)

on integrating we obtain the first solution of Lagrange's linear equation $u(x, y, z) = a$ —— ①.

Similarly choosing l', m', n' as another set of multipliers (which are not necessarily constants) we have

$$\text{Each ration} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}.$$

Such that $l'P + m'Q + n'R = 0$.

$$\Rightarrow l'dx + m'dy + n'dz = 0.$$

On integrating, we obtain the another solution of Lagrange's equation

i.e $V(x,y,z) = c_2$ — (2).

Hence the complete solution of Lagranges equation is

$\phi(u,v)=0$ or $u=f(v)$ or $v=f(u)$.

Note:- (i) Both the methods i.e Grouping and multipliers can be applied to solve the problem.

(ii) The selection of multipliers depends upon the functions P, Q, R involved in the equation. The multipliers l, m, n are called Lagrangian multipliers.

(1) Solve $x(y-z)p + y(z-x)q = z(x-y)$

Sol:- Given that $x(y-z)p + y(z-x)q = z(x-y)$.

This is a Lagranges linear equation $Pp+Qq=R$

$P = x(y-z)$ $Q = y(z-x)$ $R = z(x-y)$.

The Lagrange's auxiliary equations are.

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

By taking 1, 1, 1 as first set of multipliers, we get

$$\text{Each ratio} = \frac{1dx+1dy+1dz}{x(y-z)+y(z-x)+z(x-y)}$$

$$= \frac{dx+dy+dz}{0}$$

$dx+dy+dz = 0$ ($\because$ on cross multiplication)

Integrating

$$\int(dx+dy+dz) = c_1$$

$$x+y+z = c_1 = u(x,y,z).$$

Again by taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as another set of multipliers

we have Each ratio $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(y-z) + \frac{1}{y}y(z-x) + \frac{1}{z}z(x-y)}$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$ ($\because$ on cross multi-plication)

Integrating

$$\int \left(\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz\right) = \log c_2$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2 = v(x,y,z)$$

The complete solution of given equation is $\phi(u,v) = 0$

i.e $\phi(x+y+z, xyz) = 0$.

(2) Find the partial diff. eqn $(y+z)p + (z+x)q = x+y$.

Sol:- Given that $(y+z)p + (z+x)q = x+y$.

Here $P = y+z$ $Q = z+x$ $R = x+y$.

The Lagranges subsidary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}.$$

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(i) Taking 1, −1, 0 as multipliers, we get

Each fraction is also $= \dfrac{dx-dy}{y-x}$. — (2)

(ii) Taking 0, 1, −1 as multipliers, we get

Each fraction is also $= \dfrac{dy-dz}{z-y}$. — (3)

(iii) Taking 1, 1, 1 as multipliers, we get

Each fraction is also $= \dfrac{dx+dy+dz}{2(x+y+z)}$ — (4)

From (1), (2) (3) and (4), we get

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}.$$

Taking 4th and 5th members.

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

Integrating both sides, we get

$$\int \frac{dx-dy}{x-y} = \int \frac{dy-dz}{y-z} + \log c_1.$$

$$\log|x-y| = \log|y-z| + \log c_1.$$

$$\log\left|\frac{x-y}{y-z}\right| = \log c_1$$

$$(x-y) = c_1(y-z)$$

Taking 5th and 6th members

$$\frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

Integrating bothsides, we get

$$-2\int \frac{dy-dz}{y-z} = \int \frac{dx+dy+dz}{x+y+z} + \log c_2$$

$$-2\log|y-z| = \log|x+y+z| + \log c_2$$

$$\log\left|\frac{1}{(y-z)^2}\right| = \log|c_2(x+y+z)|$$

$$\frac{1}{c_2} = (x+y+z)(y-z)^2 .$$

$\therefore$ The solution of the given partial differential equation

$$\phi(c_1, c_2) = 0 \text{ i.e. } \phi\left(\frac{x-y}{y-z}, (y-z)^2(x+y+z)\right) = 0 .$$

Solve $(x^2-yz)p+(y^2-zx)q = z^2-xy$.

Sol:- Given that $(x^2-yz)p + (y^2-zx)q = z^2-xy$ —— ①

Compare eqn ① with $Pp+Qq=R$.

$P = x^2-yz$ $\quad Q = y^2-zx \quad R = z^2-xy$.

The Lagrange's Auxiliary equations are.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

i.e $$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}.$$

(i) Taking $1, -1, 0$ as one set of multipliers.

$$\text{Each ratio} = \frac{dx-dy}{(x^2-yz)-(y^2-zx)}.$$

$$= \frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dx-dy}{(x-y)(x+y+z)} \quad \text{—— ②}$$

(ii) Taking $0, 1, -1$ as another set of multipliers.

$$\text{Each ratio} = \frac{dy-dz}{(y^2-zx)-(z^2-xy)}$$

$$= \frac{dy-dz}{(y^2-z^2)+x(y-z)} = \frac{dy-dz}{(y-z)(x+y+z)} \quad \text{—— ③}$$

From ② and ③, we get

$$\text{Each ratio} = \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

Integrating bothsides, we get

$$\int \frac{dx-dy}{x-y} = \int \frac{dy-dz}{y-z} + \log c_1$$

$$\log|x-y| - \log|y-z| = \log c_1$$

$$\log\left|\frac{x-y}{y-z}\right| = \log c_1$$

$$\frac{x-y}{y-z} = c_1$$

(iii) Taking x, y, z as a set of multipliers, we get

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2-yz)+y(y^2-zx)+z(z^2-xy)} = \frac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad \text{--- (4)}$$

(iv) Taking $1, 1, 1$ as another set of multipliers, we get

$$\text{Each ratio} = \frac{dx+dy+dz}{(x^2-yz)+(y^2-zx)+(z^2-xy)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} \quad \text{--- (5)}$$

From (4) and (5), we get

$$\text{Each ratio} = \frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$x\,dx+y\,dy+z\,dz = (x+y+z)(dx+dy+dz)$$

Integrating bothsides, we get

$$\int x\,dx + \int y\,dy + \int z\,dz = \int (x+y+z)(dx+dy+dz) + c_2$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} - \frac{(x+y+z)^2}{2} = c_2$$

$$\frac{(x^2+y^2+z^2)-(x+y+z)^2}{2} = c_2$$

$$xy + yz + zx = -c_2$$

$$xy + yz + zx = c_3 \quad \text{where } c_3 = -c_2 .$$

$\therefore$ The complete sol. of given P.D.E is .

$$\phi\left(\frac{x-y}{y-z},\ xy + yz + zx\right)$$

Solve. $p\cos(x+y) + q\sin(x+y) = z$.

sol:- Given that $p\cos(x+y) + q\sin(x+y) = z$ —— ①

Compare equation ① with $Pp + Qq = R$.

Here $P = \cos(x+y)$ $\quad Q = \sin(x+y)$ $\quad R = z$.

The Lagrange's Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} .$$

$$\text{i.e } \frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} \quad \text{——— ②}$$

(i) Taking 1, 1, 0 as a set of multipliers, we get .

$$\text{Each ratio} = \frac{dx + dy}{\cos(x+y) + \sin(x+y)} \quad \text{—— ③}$$

(ii) Taking 1, −1, 0 as another set of multipliers, we get .

$$\text{Each ratio} = \frac{dx - dy}{\cos(x+y) - \sin(x+y)} \quad \text{— ④}$$

From ②, ③ and ④, we get .

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} = \frac{dx+dy}{\sin(x+y) + \cos(x+y)} = \frac{dx-dy}{\cos(x+y) - \sin(x+y)}$$

→ Taking 3rd and 4th members, we get .

$$\frac{dz}{z} = \frac{dx + dy}{\cos(x+y) + \sin(x+y)}$$

put $x + y = t \Rightarrow dx + dy = dt$.

$$\frac{dz}{z} = \frac{dt}{\cos t + \sin t}$$

$$\frac{dz}{z} = \frac{dt}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right)}$$

$$\sqrt{2}\,\frac{dz}{z} = \frac{dt}{\sin\frac{\pi}{4}\cos t + \cos\frac{\pi}{4}\sin t}$$

$$\sqrt{2}\,\frac{dz}{z} = \frac{dt}{\sin\left(t+\frac{\pi}{4}\right)}$$

$$\sqrt{2}\,\frac{dz}{z} = \operatorname{cosec}\left(t+\frac{\pi}{4}\right) dt.$$

Integrating bothsides, we get.

$$\sqrt{2}\int\frac{dz}{z} = \int \operatorname{cosec}\left(t+\frac{\pi}{4}\right) dt + \log c_1$$

$$\sqrt{2}\log|z| = \log\tan\frac{1}{2}\left(t+\frac{\pi}{4}\right) + \log c_1.$$

$$\log|z|^{\sqrt{2}} = \log c_1\tan\frac{1}{2}\left(t+\frac{\pi}{4}\right)$$

$$z^{\sqrt{2}} = c_1\tan\frac{1}{2}\left(t+\frac{\pi}{4}\right)$$

$$z^{\sqrt{2}}\cot\frac{1}{2}\left(x+y+\frac{\pi}{4}\right) = c_1 \quad [\because x+y=t]$$

→ Taking 4^{th} and 5^{th} members, we get

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

$$\frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}(dx+dy) = dx-dy.$$

[put $x+y=t$
$dx+dy=dt$]

$$\frac{\cos t-\sin t}{\cos t+\sin t}\,dt = dx-dy.$$

Integrating bothsides, we get.

$$\int\frac{\cos t-\sin t}{\cos t+\sin t}\,dt = \int dx - \int dy + \log c_2$$

$$\log|\cos t+\sin t| - \log c_2 = x-y$$

$$\log\left|\frac{\cos t+\sin t}{c_2}\right| = x-y.$$

$$\frac{\cos t+\sin t}{c_2} = e^{x-y}$$

$$e^{-(x-y)}(\cos t+\sin t) = c_2$$

∴ The complete sol. of given P.D.E is

$$\phi\left(z^{\sqrt{2}}\,\cancel{\cot}\,\tfrac{1}{2}\left(x+y+\tfrac{\pi}{4}\right),\ e^{y-x}(\cos t+\sin t)\right) = 0.$$

Solve $(x^3+3xy^2)p + (y^3+3x^2y)q = 2(x^2+y^2)z$.

sol: Given that $(x^3+3xy^2)p + (y^3+3x^2y)q = 2(x^2+y^2)z$ —— ①

Compare eqn ① with $Pp+Qq=R$.

Here $P = x^3+3xy^2$ $\quad Q = y^3+3x^2y$ $\quad R = 2(x^2+y^2)z$

The Lagrange's Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2z(x^2+y^2)} \quad \text{—— ②}$$

(i) choose 1, 1, 0 as one set of multipliers, we get

$$\text{Each ratio} = \frac{dx+dy}{x^3+3xy^2+y^3+3x^2y}$$

$$= \frac{dx+dy}{(x+y)^3} \quad \text{—— ③}$$

(ii) choose 1, −1, 0 as another set of multipliers, we get

$$\text{Each ratio} = \frac{dx-dy}{x^3+3xy^2-y^3-3x^2y}$$

$$= \frac{dx-dy}{(x-y)^3} \quad \text{—— ④}$$

(iii) choose $\frac{1}{x}, \frac{1}{y}, 0$ as 3^{rd} set of multiplieers, we get

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy}{\frac{1}{x}(x^3+3xy^2)+\frac{1}{y}(y^3+3x^2y)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy}{4(x^2+y^2)} \quad \text{——} \quad (5)$$

From (2), (3), (4) and (5), we get

$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2z(x^2+y^2)} = \frac{dx+dy}{(x+y)^3} = \frac{dx-dy}{(x-y)^3} = \frac{\frac{1}{x}dx+\frac{1}{y}dy}{4(x^2+y^2)}$$

→ Taking 3th and 4th members, we get.

$$\frac{dx+dy}{(x+y)^3} = \frac{dx-dy}{(x-y)^3}.$$

Integrating bothsides, we get

$$\int \frac{dx+dy}{(x+y)^3} - \int \frac{dx-dy}{(x-y)^3} = c_1.$$

$$-\frac{1}{2(x+y)^2} + \frac{1}{2(x-y)^2} = c_1.$$

→ Taking 3rd and 6th members, we get

$$\frac{dz}{2z(x^2+y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy}{4(x^2+y^2)}$$

Integrating bothsides, we get

$$\int \frac{dz}{z} - \frac{1}{2}\left[\int \frac{1}{x}dx + \int \frac{1}{y}dy\right] = \log c_2$$

$$\log|z| - \frac{1}{2}\left[\log|x| + \log|y|\right] = \log c_2$$

$$\log|z| - \frac{1}{2}\log|xy| = \log c_2$$

$$\log\left|\frac{z}{\sqrt{xy}}\right| = \log c_2$$

$$\frac{z}{\sqrt{xy}} = c_2$$

The complete sol. of given P.D.E is $\phi\left(\frac{z}{\sqrt{xy}}, \frac{1}{2(x-y)^2} - \frac{1}{2(x+y)^2}\right) = 0.$

Non Linear Partial Differential Equations of First Order :—

A Partial differential equation in which the partial derivatives p and q occur other than the first degree and are multiplied together is said to be a non linear partial differential eqn. of first order.

Complete Integral or Complete solution :—

A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete solution of the given equation.

Particular Integral :—

A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral.

Singular Integral :— Let $f(x,y,z,p,q)=0$ be a partial differential equation whose complete integral is $\phi(x,y,z,a,b)=0$ —— ①.

Differentiating ① partially w.r.t a and b and then equate to zero, we get $\frac{\partial\phi}{\partial a}=0$ —— ② and $\frac{\partial\phi}{\partial b}=0$. —— ③.

Eliminating a and b by using equations ①, ② and ③.

The eliminant of a and b is called singular integral.

Solutions of a partial differential equation :-

Consider partial differential equations of first order involving two independent variables x and y and the dependent variable z. Such an equation is of the form. $f(x, y, z, p, q) = 0$ —— ①.

In this equation, p and q may appear with first or higher powers or in product forms. If p and q appear only with first powers and there occur no products among themselves, the equation is called a linear equation; otherwise it is called a non linear equation.

Complete Solution :-

Suppose $\phi(x, y, z, a, b) = 0$ —— ② is a relation from which the partial differential equation ① is derived by eliminating the arbitrary constants a and b. Then ② is called a complete solution (integral) of equation ①.

For given values of a and b, relation ② represents a surface. As a and b are chosen arbitrary, this relation represents a two parameter family of surfaces.

Eg:- Eliminating the arbitrary constants a and b, the relation

$z = (x^2+a)(y^2+b)$ —— ① yields a partial differential equation

$pq = 4xyz$ —— ②

∴ The relation ① is a complete solution of the 1^{st} order P.D.E

Particular Solution :-

Suppose we give particular values to the arbitrary constants a and b present in the complete solution $\phi(x, y, z, a, b) = 0$ of equation $f(x, y, z, p, q) = 0$. Then $\phi(x, y, z, a, b) = 0$ becomes a particular solution (integral) of equation $f(x, y, z, p, q) = 0$. This solution represents a particular member of the family of surfaces given by the complete solution $\phi(x, y, z, a, b) = 0$.

Eg:- $z = (x^2+a)(y^2+b)$ is a complete solution of $pq = 4xyz$.
↳ ①

if we put $a=2, b=1$ in solution ①, we get $z=(x^2+2)(y^2+1)$ as a particular solution of equation $pq=4xyz$. This represents a particular surface of the family of surfaces given by the complete solution $z=(x^2+a)(y^2+b)$.

General Solution :-

Suppose in the complete solution $\phi(x,y,z,a,b)=0$ ① we take b as known function of a say $b=\psi(a)$ (or vice versa) Then ① becomes.

$$\phi(x,y,z,a,\psi(a))=0 \text{ --- } ②.$$

Diff ② partially w.r.to 'a', we get.

$$\frac{\partial\phi}{\partial a}+\frac{\partial\phi}{\partial\psi}\psi'(a)=0 \text{ --- } ③.$$

Suppose we eliminate "a" from relations ② and ③, if the elimination is possible. The resulting relation, if it satisfies $f(x,y,z,p,q)=0$, is called a general solution (integral) of equation $f(x,y,z,p,q)=0$.

This solution represents the envelope of the one parameter family of surfaces represented by $\phi(x,y,z,a,\psi(a))=0$.

For example :- $z=(x^2+a)(y^2+b)$ ① is complete solution of $pq=4xyz$ ②.

Suppose we take $b=a$ Then we get

$$z=(x^2+a)(y^2+a) \text{ --- } ③$$

$$x^2y^2+a(x^2+y^2)+a^2-z=0 \text{ --- } ④$$

Differentiating this partially, w.r.t 'a', we get

$$x^2+y^2+2a=0$$

$$\Rightarrow a=-\frac{1}{2}(x^2+y^2) \text{ --- } ⑤$$

Sub ⑤ in ④, we get.

$$x^2y^2-\frac{1}{2}(x^2+y^2)^2+\frac{1}{4}(x^2+y^2)^2-z=0.$$

$$(x^2-y^2)^2+4z=0.$$

This is a general solution of equation ②.

Singular Solution :-

Let $\phi(x,y,z,a,b)=0$ (1) be a complete solution of $f(x,y,z,p,q)=0$ (2).

Diff (1) w.r. to "a" and 'b' partially, we get

$$\frac{\partial\phi}{\partial a}=0, \quad \frac{\partial\phi}{\partial b}=0 \quad \text{--- (3)}$$

Suppose it is possible to eliminate a and b from relations (1) and (3). Then the resulting relation, if it satisfies equation (2), is called the singular solution (integral) of equation (2). This solution represents the envelope of the two-parameter family of surfaces represented by the complete solution. $\phi(x,y,z,a,b)=0$.

Eg:- $z=(x^2+a)(y^2+b)$ (1) is complete solution of $pq=4xyz$ (2).

Diff (1) w.r.t 'a' and 'b' partially, we get

$$y^2+b=0, \quad x^2+a=0.$$

Substituting for a and b from these in (1), we get

$$z=0. \quad \text{--- (3)}$$

This satisfies equation (2).

Hence this is the singular solution of equation (2).

Special types of Non linear equations of first order :-

Standard form 1 :-

An equation is of the form $f(p,q)=0$ i.e Equations involving only p and q and no x, y, z.

Let the required solution be.

$$z = ax + by + c \quad \text{--- (1)}$$

Diff (1) w.r.t x and y partially, we get

$$\frac{\partial z}{\partial x} = a \quad \text{and} \quad \frac{\partial z}{\partial y} = b$$

$$p = a \quad \text{and} \quad q = b$$

Sub. these values in $f(p,q)=0$, we get

$$f(a,b) = 0.$$

From this we can obtain b in terms of a.

Let $b = \phi(a)$.

Then (1) becomes $z = ax + \phi(a)y + c$ --- (2).

This relation contains two arbitrary constants a and c and is therefore a complete solution of equation (1).

If c is a specified function of a, say $c = \psi(a)$, then (2) becomes.

$$z = ax + \phi(a)y + \psi(a) \quad \text{--- (3)}$$

Diff (3) w.r.t 'a' partially, we get.

$$0 = x + \phi'(a)y + \psi'(a). \quad \text{--- (4)}$$

Eliminating a from (3) and (4), we get a general solution of equation (1).

Diff (2) w.r.t a and c partially, we get

$$0 = a + \psi'(a)y \quad \text{--- (5)}$$

$$0 = 1. \quad \text{--- (6)}$$

To get the singular solution, we have to eliminate a and c from relations (2), (5) and (6).

Since (6) is absurd, we infer that there is no singular solution for equation (1).

Solve the equation $q = 3p^2$. Also, find its general solution that passes through (contains) the point $(-1,0,0)$.

Sol:- Given that $q = 3p^2$ —— ①

This is of the form $f(p,q) = 0$.

The complete solution of given P.D.E is $z = ax + by + c$ —— ②

Diff ② w.r.t x and partially we get

$$\frac{\partial z}{\partial x} = a \quad \text{i.e} \quad p = a.$$

$$\frac{\partial z}{\partial y} = b \quad \text{i.e} \quad q = b.$$

Sub. p and q in ①, we get $b = 3a^2$.

$$z = ax + 3a^2y + c.$$

If this solution contains the point $(-1,0,0)$, it should be satisfied for. $x = -1, y = 0, z = 0$. that is we should have $c = a$, and the solution becomes $z = a(x+1) + 3a^2y$ —— ③.

Diff ③ w.r.t 'a' partially, we get.

$$0 = (x+1) + 6ay. \implies a = \frac{-(x+1)}{6y}.$$

Sub. a in ③, we get

$$z = -\frac{(x+1)}{6y}(x+1) + 3\left(\frac{-(x+1)}{6y}\right)^2 y.$$

$$z = \frac{-(x+1)^2}{12y}.$$

$$(x+1)^2 + 12yz = 0.$$

This is the general solution of the given equation, that passes through the point $(-1,0,0)$.

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Solve $p^2+q^2=npq$

Sol:- Given that $p^2+q^2=npq$ ——— (1)

Let the required solution be $z=ax+by+c$ ——— (2)

Diff (2) w.r.t 'x' and 'y' partially, we get

$$\frac{\partial z}{\partial x}=a \Rightarrow p=a$$

$$\frac{\partial z}{\partial y}=b \Rightarrow q=b$$

Sub. these values in (1), we get.

$$a^2+b^2-nab=0 \text{ ——— (3)}$$

To find complete integral, we have to eliminate any one of the arbitrary constants

From (3). We have $b^2-nab+a^2=0$.

Which is a quadratic equation in b.

$$\therefore\ b=\frac{na\pm\sqrt{n^2a^2-4a^2}}{2}$$

$$=\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right] \text{ ——— (4).}$$

Sub. (4) in (1), we get.

$$z=ax+\frac{ay}{2}\left[n\pm\sqrt{n^2-4}\right]+c.$$

This is the complete integral because it contains only two arbitrary constants.

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Standard form 2 :-

Equation of the form $f(z,p,q)=0$ i.e Equations containing z, p, q only and no x, y :

Let us assume that z is a function of u and $u = x+ay$.

$$z = f(u) \quad \text{and} \quad u = x+ay .$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} .$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du} .$$

[$\because$ z is a function of single independent variable u, so we use ordinary derivative $\frac{dz}{du}$ and u is a function of x, y. so we use partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$].

Substituting these values of p and q in the given equation, we get $f\left(z, \frac{dz}{du}, a\frac{dz}{\partial u}\right)=0$

Which is an ordinary differential equation of the first order and it can be solved.

Finally, replace $u = x+ay$ to get the complete solution of the given equation.

Working Procedure :—

Step 1 :- Assume $z = f(u)$ and $u = x+ay$ so that

$$p = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du} .$$

Step 2 :- Substitute the values of p and q in the given equation

Step 3 :- Solve the resulting ordinary differential equation in z and u

Step 4 :- Replace $u = x+ay$ in the complete solution.

Note :- To solve an equation of the above form, we also assume that $z = f(u)$ and $u = y+ax$ Then $p = \frac{\partial z}{\partial x} = a\frac{dz}{du}$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}.$$

(1) Solve $z^2 = 1+p^2+q^2$.

Sol:- Given that the equation of the form $f(z,p,q) = 0$.

Let $z=f(u)$ where $u = x+ay$ be the solution of the given equation.

Then $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$.

Substituting these values of p and q in the given equation, we have.

$$z^2 = 1+\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$z^2-1 = \left(\frac{dz}{du}\right)^2(1+a^2)$$

$$\sqrt{z^2-1} = \frac{dz}{du}\sqrt{1+a^2}$$

Separating the variables and integrating both sides, we get

$$\int\frac{dz}{\sqrt{z^2-1}} = \frac{1}{\sqrt{1+a^2}}\int du + c.$$

$$\cosh^{-1} z = \frac{u}{\sqrt{1+a^2}} + c.$$

Hence the solution is.

$$\cosh^{-1}(z) = \frac{x+ay}{\sqrt{1+a^2}} + c.$$

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Verify that $z = ae^{b(x+by)}$ is a complete solution of the equation $p^2 = qz$. show that $z=0$ is both a particular solution and the singular solution.

Sol:- Given that $z = ae^{b(x+by)}$ ——— (1)

Diff (1) w.r.t "x" and "y" partially, we get.

$$p = \frac{\partial z}{\partial x} = ab\,e^{b(x+by)} = bz \text{ ——— (2)}$$

$$q = \frac{\partial z}{\partial y} = ab^2 e^{b(x+by)} = b^2 z \text{ ——— (3)}$$

From (2) and (3)

$$qz = b^2 z^2 = (bz)^2$$

$$qz = p^2$$

$$\therefore \; p^2 = qz \text{ ——— (4).}$$

$p^2 = qz$ is a P.D.E obtained by eliminating the constants a and b from the relation. In other words (1) is a Complete solution of equation (4).

For $a=0$, the complete solution (1) becomes $z=0$.

Thus, $z=0$ is a particular solution.

Diff (1) w.r.t 'a' and 'b' partially, we get

$$0 = e^{b(x+by)} \text{ —— (5)}$$

$$0 = a\,e^{b(x+by)}(x+2by) \text{ —— (6).}$$

Relations (1), (5) and (6) yield $z=0$. This obeys equation (4).

Thus $z=0$ is the singular solution also.

Standard form 3 :-

Equation of the form $f_1(x,p) = f_2(y,q)$.

i.e Equation in which z is absent and the term involving x and p can be separated from those involving y and q.

Let $f_1(x,p) = f_2(y,q) = a$ [Constant]

Now solve these equations for p and q by taking $f_1(x,p) = a$ and $f_2(y,q) = a$.

Solving for p and q, we obtain.

$$p = F_1(x,a) \quad \text{and} \quad q = F_2(y,a)$$

Since $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$dz = p\,dx + q\,dy$$

Sub. the values of p and q in above equation.

$$dz = F_1(x,a)\,dx + F_2(y,a)\,dy.$$

Integrating both sides, we get

$$\int dz = \int F_1(x,a)\,dx + \int F_2(y,a)\,dy + c$$

$$z = \int F_1(x,a)\,dx + \int F_2(y,a)\,dy + c.$$

Which is the required complete solution of the given equation.

1 Solve $pq + qx = y$

Sol:- Given that $pq + qx + y$.

This is of the form $f(p, q, x, y) = 0$.

The given equation can be written as.

$$q(p+x) = y \quad \text{i.e} \quad p+x = \frac{y}{q}.$$

Let $p + x = \frac{y}{q} = a$ (constant) Then.

$p + x = a$ and $\frac{y}{q} = a$

$p = a - x \qquad q = \frac{y}{a}$.

We know that $dz = p\,dx + q\,dy$

$$dz = (a-x)\,dx + \frac{y}{a}\,dy.$$

Integrating bothsides, we get

$$\int dz = \int (a-x)\,dx + \frac{1}{a}\int y\,dy + c.$$

$$z = ax - \frac{x^2}{2} + \frac{1}{a}\frac{y^2}{2} + c.$$

$$2az = a^2x - ax^2 + y^2 + 2ac.$$

$2az = a^2x - ax^2 + y^2 + c_1$ where $c_1 = 2ac$.

Which is the complete integral.

Solve the equation $z^2(p^2z^2+q^2)=1$. Show that this equation has no singular solution.

Sol:- Given that $z^2(p^2z^2+q^2)=1$ —— ①.

This is of the form $f(p,q,z)=0$.

Let $z=f(u)$, $u=x+ay$ be the complete solution of ①

$$p=\frac{\partial z}{\partial x}=\frac{dz}{du}\frac{\partial u}{\partial x}=\frac{dz}{du}.$$

$$q=\frac{\partial z}{\partial y}=\frac{dz}{du}\frac{\partial u}{\partial y}=a\frac{dz}{du}$$

$u=x+ay$

$\frac{\partial u}{\partial x}=1$

$\frac{\partial u}{\partial y}=a$.

Sub. p and q in ①, we get

$$z^2\left(\frac{dz}{du}\right)^2(z^2+a^2)=1$$

$$\frac{dz}{du}=\pm\frac{1}{z\sqrt{z^2+a^2}}$$

$$\pm z\sqrt{z^2+a^2}\,dz=du.$$

Integrating bothsides, we get

$$\pm\int z\sqrt{z^2+a^2}\,dz=\int du+b.$$

$$\pm\frac{(z^2+a^2)^{3/2}}{3}=u+b.$$

$$(z^2+a^2)^{3/2}=3(x+ay+b)$$

$$(z^2+a^2)^3=9(x+ay+b)^2 \quad \text{——} \ (*)$$

This is a complete solution of the given P.D.E —— ②.

Diff ② w.r.t a and b partially, we get

$$18(x+ay+b)y=6(z^2+a^2)^2a \quad \text{——} \ ③$$

$$18(x+ay+b)=0 \quad \text{——} \ ④.$$

From these, we get $a=0$ and $x+ay+b=0$.

Putting these in $(*)$, we get $z=0$.

We check that $z=0$ does not satisfy the given equation.

$\therefore$ $z=0$ is not the singular solution.

Thus, the given equation has no singular solution.

Solve the equation $q^2 = p^2 z^2 (1-p^2)$. Show that $z=0$ is the singular solution.

Sol:- Given that $f(z) = q^2 = p^2 z^2 (1-p^2)$ —— ①.

This is of the form $f(p,q,z)=0$.

Let $z = f(u)$ where $u = x+ay$ be the complete sol. of ①

$u = x+ay$

$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a.$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = a\frac{dz}{du}.$$

Sub. p and q in ①, we get.

$$\left(a\frac{dz}{du}\right)^2 = \left(\frac{dz}{du}\right)^2 z^2 \left(1-\left(\frac{dz}{du}\right)^2\right).$$

$$a^2 = z^2\left(1-\left(\frac{dz}{du}\right)^2\right)$$

$$\frac{dz}{du} = \pm\frac{\sqrt{z^2-a^2}}{z}$$

$$\pm\frac{z\,dz}{\sqrt{z^2-a^2}} = du.$$

Integrating bothsides, we get

$$\pm\frac{1}{2}\int\frac{2z\,dz}{\sqrt{z^2-a^2}} = \int du + b.$$

$$\pm\sqrt{z^2-a^2} = u+b.$$

$$z^2-a^2 = (x+ay+b)^2 \quad \text{—— ②}.$$

This is the complete solution of the given equation.

Diff ② w.r.t a and b partially, we get

$$-2a = 2(x+ay+b)y, \quad 0 = 2(x+ay+b)$$

These relations give $x+ay+b=0$ and $a=0$.

Putting these in ①, we get $z=0$.

Which satisfies the given relation.

∴ $z=0$ is the singular solution of the given equation.

Standard form IV Clairaut's Form :—

Equations of the form $z = px + qy + f(p,q)$. Is called Clairaut's equation

The complete solution of the equation $z = px + qy + f(p,q)$ is

$z = ax + by + f(a,b)$.

Let the required solution be $z = ax + by + c$ Then.

$p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$ and putting for a, b in ①.

Note:- For the solution of $z = px + qy + f(p,q)$ replace p by a and q by b.

(1) Prove that the complete integral of $z = px + qy + \sqrt{p^2+q^2+1}$ represents all planes at unit distance from the origin.

Sol:- The given equation is of the form $z = px + qy + f(p,q)$.

A Complete solution is $z = ax + by + \sqrt{a^2+b^2+1}$.

or $ax + by - z + \sqrt{a^2+b^2+1} = 0$. —①.

Which is a family of planes.

The length of the perpendicular drawn from the origin to the planes ①

$$= \frac{\sqrt{a^2+b^2+1}}{\sqrt{a^2+b^2+1}} = 1.$$

Verify that $z = ax^2 + by^2 + ab$ is a complete solution of the equation $pq + 2xy(px+qy) = 4xyz$ and show that $z = 1 + 2xy$ is the general solution containing the point $(0,0,1)$.

Sol. Given that $z = ax^2 + by^2 + ab$ —— ①.

Diff ① w.r.t x and y partially, we get

$$p = \frac{\partial z}{\partial x} = 2ax, \quad q = \frac{\partial z}{\partial y} = 2by$$

Sub. for a and b from these in ①, we get

$$z = \left(\frac{p}{2x}\right)x^2 + \left(\frac{q}{2y}\right)y^2 + \frac{pq}{4xy}$$

$$pq + 2xy(px+qy) = 4xyz \quad \text{——} ②$$

This is a P.D.E obtained by eliminating the arbitrary constants a and b from the relation ①.

In other words, ① is a complete solution of equation ②.

If the solution ① contains the point $(0,0,1)$, we get $ab=1$.

Which gives $b = \frac{1}{a}$.

$$z = ax^2 + \frac{y^2}{a} + 1 \quad \text{——} ③$$

Diff. this partially w.r.t 'a', we obtain

$$0 = x^2 - \frac{y^2}{a^2}$$

So that $a = \frac{y}{x}$

Putting this in ③, we get $z = 1 + 2xy$.

This is the general solution of ② containing the point $(0,0,1)$.

Standard form IV Clairaut's Form :-

Equations of the form $z = px + qy + f(p,q)$ is called clairauts eqn. —— ①

The complete sol. of the equation $z = px + qy + f(p,q)$ is

$z = ax + by + f(a,b)$. —— ②

Let the required solution be $z = ax + by + c$ Then.

$p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$ and putting for a, b in ①.

Note :- For the solution of $z = px + qy + f(p,q)$ replace p by a and q by b.

→ When b is a specified fun. of a say $b = \phi(a)$, the solution ② becomes $z = ax + \phi(a)y + f(a, \phi(a))$. —— ③

Diff. ③ w.r.t 'a', partially, we get.

$$0 = x + \phi'(a)y + f'(a) \quad \text{—— ④}$$

Eliminating a between ③ and ④. we get a general sol. of the given equation.

→ Diff ② w.r.to a and b partially, we get.

$$0 = x + \frac{\partial f}{\partial a}, \quad 0 = y + \frac{\partial f}{\partial y} \quad \text{—— ⑤}$$

Eliminating a and b from these relations and ②, we get the singular sol. of the given equation.

(1) Prove that the complete integral of $z = px + qy + \sqrt{p^2+q^2+1}$ represents all planes at unit distance from the origin.

Sol: Given that $z = px + qy + \sqrt{p^2+q^2+1}$

The given equation is of the form $z = px + qy + f(p,q)$.

A ~~Eo~~ Which is the clairaut's equation.

A complete sol. of given equation is obtained by replacing p by a and q by b, we get.

$$z = ax + by + \sqrt{a^2+b^2+1}.$$

$$\text{(or)} \quad ax + by - z + \sqrt{a^2+b^2+1} = 0. \quad \text{——} (1)$$

Which represents a family of planes :

The length of the ~~per~~ drawn from the origin to the planes (1) is.

$$= \frac{\sqrt{a^2+b^2+1}}{\sqrt{a^2+b^2+1}} = 1.$$

Find the complete solution of the equation $pqz = p^2(qx+p^2) + q^2(py+q^2)$

Sol:- Given that $pqz = p^2(qx+p^2) + q^2(py+q^2)$

$$pqz = p^2qx + p^4 + q^2py + q^4.$$

$$z = p\frac{x}{\cancel{p}} + \frac{p^3}{q} + qy + \frac{q^3}{p}.$$

$$z = px + qy + \left(\frac{p^3}{q} + \frac{q^3}{p}\right)$$

Which is of the form $z = px + qy + f(p,q)$.

Which is the clairaut's equation..

A complete sol. of given equation is obtained by replacing p by a q by b, we get.

$$z = ax + by + \left(\frac{a^3}{b} + \frac{b^3}{a}\right).$$

Solve $z = px + qy + p^2 - q^2$.

Sol:- Given that $z = px + qy + p^2 - q^2$.

Which is in the clairaut's form.

The complete solution is $z = ax + by + a^2 - b^2$ ——(1).

General Integral:

Put $b = \phi(a)$ in (1), we get

$z = ax + \phi(a)y + a^2 - [\phi(a)]^2$ ——(2).

Diff (2) w.r.t 'a', we get

$0 = x + \phi'(a) \cdot y + 2a - 2\phi(a)\phi'(a)$ ——(3).

Eliminating a between (2) and (3), we get the general integral.

Singular Integral:

Diff (1) w.r.t a and b.

$0 = x + 2a. \Rightarrow a = \frac{-x}{2}$.

$0 = y - 2b \Rightarrow b = \frac{y}{2}$.

Sub. in (1), $z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2 - y^2}{4}$.

$4z = y^2 - x^2$. is the singular solution.

Solve $z = px + qy + p^2 q^2$.

Sol:- Given that $z = px + qx + p^2 q^2$.

Which is clairaut's form.

The complete solution is $z = ax + by + a^2 b^2$ ——(1).

Singular Integral:-

Diff (1) w.r.t a and b.

$0 = x + 2ab^2 \Rightarrow x = -2ab^2$

$0 = y + 2b^2 a. \Rightarrow y = -2b^2 a$.

$\frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{c}$ say.

$a = cy, \quad b = cx$.

Sub. in ②, we get

$$x = -2c^3yx^2$$

$$c^3 = \frac{-1}{2xy}$$

Sub. a and b in ①.

$$z = cxy + cxy + c^4x^2y^2.$$

$$z = 2cxy + cx^2y^2\left(\frac{-1}{2xy}\right)$$

$$z = 2cxy - \frac{c}{2}xy = \frac{3}{2}cxy$$

$$z^3 = \frac{27}{8}c^3x^3y^3.$$

$$z^3 = \frac{27}{8}x^3y^3\left(\frac{-1}{2xy}\right)$$

$$z^3 = -\frac{27}{16}x^2y^2.$$

$16z^3 + 27x^2y^2 = 0$ is the singular integral.

General Integral :-

Put $b = \phi(a)$ in ①.

$$z = ax + \phi(a)y + a^2[\phi(a)]^2 \text{ --- } ③.$$

Diff ③ w.r.t a and eliminating a, the general integral can be obtained.

Solve $z = px + qy + \log pq$.

Sol: Given that $z = px + qy + \log pq$.

This is clairaut's form.

The complete sol. is $z = ax + by + \log ab$ ——①

Singular integral Diff ① w.r.t a and b partially, we get

$$0 = x + \frac{1}{a} \Rightarrow a = -\frac{1}{x}, \quad 0 = y + \frac{1}{b} \Rightarrow b = -\frac{1}{y}.$$

Sub. in ① the singular integral is $z = -1 - 1 + \log\left(\frac{1}{xy}\right)$

$$z = -2 - \log xy.$$

General Integral :- Put $b = \phi(a)$ in ①.

$$z = ax + \phi(a)y + \log a\phi(a) \text{ --- } ②$$

Diff ② w.r.t a and eliminating a the general integral can be obtained.

Solve $z = px + qy - 2\sqrt{pq}$.

Sol: Given that $z = px + qy - 2\sqrt{pq}$.

This is in clairaut's form

The complete integral is $z = ax + by - 2\sqrt{ab}$ —— ①.

Singular Integral :-

Diff ① w.r.t a and b partially, we get.

$$0 = x - \frac{2}{2\sqrt{ab}}\, b \quad \text{i.e} \quad x = \sqrt{\frac{b}{a}}$$

$$0 = y - \frac{2}{2\sqrt{ab}}\, a \quad \text{i.e} \quad y = \sqrt{\frac{a}{b}}$$

Eliminating a and b, the singular integral is $xy = 1$.

General Integral :-

Let $b = \phi(a)$ where ϕ is an arbitrary function.

Sub. in ①, we get $z = ax + \phi(a)y - 2\sqrt{a\phi(a)}$ —— ③.

Diff ③ w.r.t 'a', we get.

$$0 = x + \phi'(a)y - 2\left[\sqrt{a}\,\frac{1}{2}[\phi(a)]^{-\frac{1}{2}}\phi'(a) + [\phi(a)]^{1/2}\,\frac{1}{2}\,a^{-1/2}\right]$$

$$0 = x + y\phi'(a) - \left[\sqrt{a}\,\frac{\phi'(a)}{\sqrt{\phi(a)}} + \frac{\sqrt{\phi(a)}}{\sqrt{a}}\right]$$

The general integral can be obtained by eliminating 'a' from ③ and ④.

Solve $z = px + qy + c\sqrt{1+p^2+q^2}$.

Sol: Given that $z = px + qy + c\sqrt{1+p^2+q^2}$.

Which is in clairaut's form.

The Complete Integral is $z = ax + by + c\sqrt{1+a^2+b^2}$ —— ①.

Singular Integral :-

Diff ① w.r.t a and b. partially, we get.

$$0 = x + \frac{ca}{\sqrt{1+a^2+b^2}} \quad \text{—— ②}$$

$$0 = y + \frac{cb}{\sqrt{1+a^2+b^2}} \quad \text{—— ③}$$

$$x^2+y^2 = \frac{c^2(a^2+b^2)}{1+a^2+b^2}$$

$$c^2-(x^2+y^2) = c^2 - \frac{c^2(a^2+b^2)}{1+a^2+b^2} = \frac{c^2}{1+a^2+b^2}$$

$$1+a^2+b^2 = \frac{c^2}{c^2-(x^2+y^2)}$$

$$\frac{1}{\sqrt{1+a^2+b^2}} = \frac{\sqrt{c^2-(x^2+y^2)}}{c}$$

$$x = \frac{-ac}{\sqrt{1+a^2+b^2}} = \frac{-ac\sqrt{c^2-x^2-y^2}}{c}$$

$$x = -a\sqrt{c^2-x^2-y^2}$$

$$a = \frac{-x}{\sqrt{c^2-x^2-y^2}}$$

Similarly $b = \dfrac{-y}{\sqrt{c^2-x^2-y^2}}$

Substituting these values of a and b in ①, we get

$$z = \frac{-x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$z = \sqrt{c^2-x^2-y^2}$$

$$x^2+y^2+z^2 = c^2$$

This is the singular Integral.

General Integral :

Put $b=\phi(a)$ in ①.

$$z = ax+\phi(a)y + c\sqrt{1+a^2+[\phi(a)]^2} \quad \text{—④}$$

Diff ④ w.r.t a and eliminating a. general integral can be obtained.

Verify that $z = ax^2 + by^2 + ab$ is a complete sol. of the equation $pq + 2xy(px + qy) = 4xyz$ and show that $z = 1 + 2xy$ is the general solution containing the point $(0,0,1)$.

Sol: Given that $z = ax^2 + by^2 + ab$ —— ①

Diff ① w.r.t x and y partially, we get.

$$p = \frac{\partial z}{\partial x} = 2ax, \quad q = \frac{\partial z}{\partial y} = 2by.$$

$$a = \frac{p}{2x} \qquad b = \frac{q}{2y}.$$

Sub. a and b ~~from~~ in eqn ①, we get

$$z = \left(\frac{p}{2x}\right)x^2 + \left(\frac{q}{2y}\right)y^2 + \frac{pq}{4xy}$$

$$z = \frac{px}{2} + \frac{qy}{2} + \frac{pq}{4xy}$$

$$4xyz = 2px^2y + 2qxy^2$$

$$4xyz = 2xy(px + qy). \text{ —— ②}$$

This is a P.D.E obtained by eliminating the arbitrary constants a and b from the relation ①.

In otherwords, ① is a complete sol. of eqn ②.

If the sol. ① contains the point $(0,0,1)$, we get $ab = 1$.

which gives $b = \frac{1}{a}$.

$$z = ax^2 + \frac{y^2}{a} + 1 \text{ —— ③}$$

Diff ③ w.r.t 'a' partially, we get

$$0 = x^2 - \frac{y^2}{a^2}$$

so that $a = \frac{y}{x}$ —— ④

Sub. ④ in ③, we get $z = 1 + 2xy$.

This is the general sol. of ② containing the point $(0,0,1)$.

Equations Reducible to standard forms :—

Type 1 :- Equations of the type $f(x^m p, y^n q)=0$ where m and n are constants : —— ①

The above form of the equation can be transformed to an equation of the form $f(P, Q)=0$ by the substitution.

Case (i) :- When $m \neq 1$ and $n \neq 1$

Put $X = x^{1-m}$ and $Y = y^{1-n}$ then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x}$$

$$X = x^{1-m}$$

$$\frac{\partial X}{\partial x} = (1-m)\, x^{-m}.$$

$$p = P(1-m)x^{-m}$$

$$x^m p = (1-m)P$$

Where $P = \frac{\partial z}{\partial X}$.

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$$

$$Y = y^{1-n}$$

$$\frac{\partial Y}{\partial y} = (1-n)\, y^{-n}$$

$$q = Q(1-n)y^{-n}$$

Where $Q = \frac{\partial z}{\partial Y}$

$$y^n q = Q(1-n)$$

Now the given equation ① reduces to $f((1-m)P, (1-n)Q)=0$.

Which is of the form $f(P, Q)=0$. (Standard form – 1) —— ②

Let the complete sol. of P.D.E ② be of the form

$$Z = aX + bY + c \quad \text{—— ③}$$

Diff. ③ w.r.t X and Y partially, we get

$$P = \frac{\partial z}{\partial X} = a, \quad Q = \frac{\partial z}{\partial Y} = b.$$

Sub. P and Q in ②, we get $f(a,b)=0$ —— ④

Using ④, Write b interms of a i.e $b = \phi(a)$ —— ⑤

Sub. ⑤ in ③, we get $z = aX + \phi(a)Y + c$. —— ⑥

Finally replace X with x^{1-m} and Y with y^{1-n} in ⑥, we get the required complete solution of ①.

Case(ii) When $m=1$ and $n=1$

The given P.D.E is of the form $f(x^m p, y^n q)=0$.

i.e $f(xp, yq)=0$ —— ①.

Put $X=\log x$ and $Y=\log y$ then.

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}$$

$$X=\log x$$
$$\frac{\partial X}{\partial x}=\frac{1}{x}.$$

$$p=P\cdot\frac{1}{x}$$

Where $\frac{\partial z}{\partial X}=P$.

$$xp=P.$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}.$$

$$Y=\log y$$
$$\frac{\partial Y}{\partial y}=\frac{1}{y}.$$

$$q=Q\cdot\frac{1}{y}$$

Where $\frac{\partial z}{\partial Y}=Q$.

$$yq=Q.$$

Now the given equation ① reduces to $f(P,Q)=0$ —— ②.
(standard form − 1)

Let the complete sol. of P.D.E ② of the form $z=aX+bY+c$ —— ③

Diff ③ w.r.t X and Y partially, we get.

$$P=\frac{\partial z}{\partial X}=a \qquad Q=\frac{\partial z}{\partial Y}=b.$$

Sub. P and Q in ③, we get $f(a,b)=0$ —— ④

Using ④, Write b interms of a i.e $b=\phi(a)$ —— ⑤.

Sub. ⑤ in ③, we get $z=aX+\phi(a)Y+c$ —— ⑥.

Finally replace X with $\log x$ and Y with $\log y$ in ⑥, we get the required complete solution of ①.

Equations Reducible to standard forms :—

(1)

(i) Equations of the type $f(x^m p, y^n q) = 0$ where m and n are constants.

The above form of the equation can be transformed to an equation of the form $f(P, Q) = 0$ by the substitution.

Case (i) :- When $m \neq 1$ and $n \neq 1$

Put $X = x^{1-m}$ and $Y = y^{1-n}$ then.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P(1-m)\, x^{-m} \quad \text{where } P = \frac{\partial z}{\partial X}$$

$$x^m p = P(1-m).$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = Q(1-n) y^{-n} \quad \text{where } Q = \frac{\partial z}{\partial Y}$$

$$y^n q = Q(1-n)$$

Now the given equation reduces to $f[(1-m)P, (1-n)Q] = 0$.

Which is of the form $f(P, Q) = 0$.

Case (ii) :- When $m = 1$ and $n = 1$

Put $X = \log x$ and $Y = \log y$ Then.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{1}{x}.$$

$$px = P \quad \text{where } P = \frac{\partial z}{\partial X}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{1}{y}$$

$$qy = Q \quad \text{where } Q = \frac{\partial z}{\partial Y}.$$

Now the given equation reduces to the form $f(P, Q) = 0$

(1) Solve $px + qy = 1$.

Sol:- Given that $px + qy = 1$ ——— ①.

This is of the form $f(x^m p, y^n q) = 0$.

Here $m=1$, $n=1$.

Put $\log x = X$ and $\log y = Y$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P \cdot \frac{1}{x}$$

$$P = px. \quad \text{——— ②}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = Q \cdot \frac{1}{y}$$

$$Q = qy \quad \text{——— ③}$$

Sub. ② and ③ in ①, we get

$$P + Q = 1 \quad \text{——— ④}$$

Eqn ④ is of the form $f(P, Q) = 0$.

The complete sol. of $f(P, Q) = 0$ is $z = aX + bY + c$ ——— ⑤

Diff ⑤ w.r.t X and Y partially, we get

$$\frac{\partial z}{\partial X} = a \quad \text{i.e} \quad P = a.$$

$$\frac{\partial z}{\partial Y} = b \quad \text{i.e} \quad Q = b.$$

Sub. the values of P and Q in ④, we get

$$a + b = 1.$$

$$b = 1 - a.$$

$\therefore$ The general solution of ① is

$$z = aX + (1-a)Y + c.$$

$$z = a\log x + (1-a)\log y + c.$$

Solve $\frac{x^2}{p} + \frac{y^2}{q} = 2$.

Sol:- Given that $\frac{x^2}{p} + \frac{y^2}{q} = 2$

$$x^2 p^{-1} + y^2 q^{-1} = 2$$

$$(x^{-2}p)^{-1} + (y^{-2}q)^{-1} = 1 \quad \text{——— (1)}$$

This is of the form $f(x^m p, y^n q) = 0$.

Here $m = -2$, $n = -2$.

Put $X = x^{1-m}$ $\qquad Y = y^{1-n}$

$X = x^{1-(-2)} = x^3$ $\qquad Y = y^{1-(-2)} = y^3$

$\frac{\partial X}{\partial x} = 3x^2$ $\qquad \frac{\partial Y}{\partial y} = 3y^2$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P \cdot 3x^2 \implies x^{-2}p = 3P \text{ ——— (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = Q \cdot 3y^2 \implies y^{-2}q = 3Q \text{ ——— (3)}$$

Sub. (2) and (3) in (1), we get.

$$(3P)^{-1} + (3Q)^{-1} = 1$$

$$\frac{1}{P} + \frac{1}{Q} = 3 \quad \text{——— (4)}$$

This is of the form $f(P, Q) = 0$ (standard form-I)

Let the complete solution of (4) is $z = aX + bY + c$.

$\frac{\partial z}{\partial X} = a$ i.e $P = a$ ——— (5)

$\frac{\partial z}{\partial Y} = b$ i.e $Q = b$ ——— (6)

Sub. (5) & (6) in (4), we get.

$$\frac{1}{a} + \frac{1}{b} = 3 \implies \frac{1}{b} = \left(3 - \frac{1}{a}\right)$$

$$\frac{1}{b} = \frac{3a-1}{a} \implies b = \frac{a}{3a-1}.$$

$\therefore \quad z = ax + \frac{a}{3a-1} y + c.$

$\therefore$ The Complete solution of given P.D.E is

$z = ax^3 + \frac{a}{3a-1} \cdot y^3 + c$. where a, c are arbitrary constants.

Equations of the type $f(x^m p, y^n q, z) = 0$ where m and n are constants

This can be reduced to an equation of the form $f(P, Q, z) = 0$ by the substitutions given for the equation $f(x^m p, y^n q) = 0$.

(1) Solve $x^2p^2 + y^2q^2 = z^2$.

Sol:- Given that $x^2p^2 + y^2q^2 = z^2$.

It can be written as $(xp)^2 + (yq)^2 = z^2$. —— ①.

This is of the form $f(z, x^m p, y^n q) = 0$.

Here $m = 1$, $n = 1$.

Put $X = \log x$ and $Y = \log y$ Then,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P \cdot \frac{1}{x} \quad \text{where } P = \frac{\partial z}{\partial X}$$

$$P = px \quad \text{—— ②}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial y} = Q \cdot \frac{1}{y} \quad \text{where } Q = \frac{\partial z}{\partial Y}$$

$$Q = qy. \quad \text{—— ③}$$

Sub. ② and ③ in ①, we get

$$P^2 + Q^2 = z^2 \quad \text{—— ④}$$

Eqn ④ is of the form $f(z, P, Q) = 0$.

Let $z = f(u)$ where $u = X + aY$ be the solution ④

Then $P = \frac{dz}{du}$ and $Q = a\frac{dz}{du}$.

$$\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2.$$

$$\left(\frac{dz}{du}\right)^2 (1 + a^2) = z^2.$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}.$$

Separate the variables and integrate.

$$\int \frac{dz}{z} = \frac{1}{\sqrt{1+a^2}} \int du + c.$$

$$\log z = \frac{u}{\sqrt{1+a^2}} + c.$$

$\therefore$ The complete Integral is

$$\log z = \frac{X+aY}{\sqrt{1+a^2}} + c.$$

$$\log z = \frac{\log x + a\log y}{\sqrt{1+a^2}} + c.$$

(2) Solve $2x^4p^2 - yzq - 3z^2 = 0$.

Sol:- Given that $2x^4p^2 - yzq - 3z^2 = 0$

It can be written as $2(x^2p)^2 - yzq - 3z^2 = 0$ —— ①

This is of the form $f(z, x^m p, y^n q) = 0$.

Here $m=2$, $n=1$.

Put $X = x^{1-2} = x^{-1}$ and $Y = \log y$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\left(-\frac{1}{x^2}\right)$$

$$-P = px^2. \quad \text{—— ②}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = Q.\frac{1}{y}$$

$$Q = qy \quad \text{—— ③}$$

Sub ② and ③ in ①, we get

$$2P^2 - Qz - 3z^2 = 0 \quad \text{—— ④}$$

This is of the form $f(z, P, Q) = 0$ —— ⑤.

Let $z = f(u)$ where $u = X + aY$ be the sol. of ④.

Then $P = \frac{dz}{du}$ and $Q = a\frac{dz}{du}$ —— ⑤.

sub ⑤ in ④, we get.

$$2\left(\frac{dz}{du}\right)^2 - a\frac{dz}{du}z - 3z^2 = 0.$$

$$2\left(\frac{dz}{du}\right)^2 - az\frac{dz}{dy} - 3z^2 = 0$$

This is a quadratic equation in $\frac{dz}{du}$.

$$\frac{dz}{du} = \frac{az \pm \sqrt{a^2z^2 + 24z^2}}{4}$$

$$\frac{dz}{du} = \frac{z\left[a \pm \sqrt{a^2+24}\right]}{4}$$

Separate the variables and Integrate

$$\int\frac{dz}{z} = \frac{a \pm \sqrt{a^2+24}}{4}\int du + c.$$

$$\log z = \frac{a \pm \sqrt{a^2+24}}{4}u + c.$$

The general Sol. is.

$$\log z = \frac{a \pm \sqrt{a^2+24}}{4}(x+ay) + c.$$

$$\log z = \frac{a \pm \sqrt{a^2+24}}{4}\left(\frac{1}{x} + a\log y\right) + c.$$

(2) solve $z^2(p^2+q^2) = x^2+y^2$.

Sol:- Given that $z^2(p^2+q^2) = x^2+y^2$.

The given diff. eqn. can be written as $(pz)^2 - x^2 = y^2 - (qz)^2$ —— ①.

This is of the form $f(x, pz^n) = g(y, qz^n)$ with $n=1$.

Put $Z = z^{n+1}$

$Z = z^{1+1} = z^2$.

$Z = z^2$.

$$\frac{\partial Z}{\partial x} = 2z \frac{\partial z}{\partial x}$$

$P = 2zp$

$zp = \frac{P}{2}$.

$$\frac{\partial Z}{\partial y} = 2z \frac{\partial z}{\partial y}$$

$Q = 2zq$

$zq = \frac{Q}{2}$.

Sub. the values of zp and zq in ①, we get.

$$\left(\frac{P}{2}\right)^2 - x^2 = y^2 - \left(\frac{Q}{2}\right)^2$$

$$\frac{P^2}{4} - x^2 = y^2 - \frac{Q^2}{4}.$$

This is of the form $f_1(x, P) = f_2(y, Q)$.

$$\frac{P^2}{4} - x^2 = y^2 - \frac{Q^2}{4} = a.$$

$$\frac{P^2}{4} - x^2 = a \qquad y^2 - \frac{Q^2}{4} = a.$$

$P^2 = 4a + 4x^2 \qquad Q^2 = 4y^2 - 4a.$

$P = 2\sqrt{x^2+a}. \qquad Q = 2\sqrt{y^2-a}$

We have $dz = P\,dx + Q\,dy$

$$dz = 2\sqrt{x^2+a}\,dx + 2\sqrt{y^2-a}\,dy.$$

Integrating bothsides, we get

$$\int dz = 2\int\sqrt{x^2+a}\,dx + 2\int\sqrt{y^2-a}\,dy + c.$$

$$z = 2\left[\frac{x}{2}\sqrt{x^2+a} + \frac{a}{2}\sinh^{-1}\left(\frac{x}{\sqrt{a}}\right)\right] +$$
$$2\left[\frac{y}{2}\sqrt{y^2-a} - \frac{a}{2}\cosh^{-1}\left(\frac{y}{a}\right)\right] + c.$$

$$z = x\sqrt{x^2+a} + a\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2-a} - a\cosh^{-1}\left(\frac{y}{a}\right) + c.$$

$\therefore$ The complete solution of (1) is

$$z^2 = x\sqrt{x^2+a} + y\sqrt{y^2-a} + a\sinh^{-1}\left(\frac{x}{a}\right) - a\cosh^{-1}\left(\frac{y}{a}\right) + c.$$

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Equations of the type $f_1(x, pz^n) = f_2(y, qz^n)$ where n is constant

An equation of the above form can be reduced to an equation of the form $f_1(x, P) = f_2(y, Q)$ by the substitutions

Put $Z = \begin{cases} z^{n+1} & \text{if } n \neq -1 \\ \log z & \text{if } n = -1 \end{cases}$

1) solve $(x+pz)^2 + (y+qz)^2 = 1.$

Sol:- Given that $(x+pz)^2 + (y+qz)^2 = 1.$

The given diff. eqn can be written as

$$(x+pz)^2 = 1-(y+qz)^2$$

This is of the form $f(x, pz^n) = g(y, qz^n)$ with $n=1$

Put $Z = z^{n+1}$

$$Z = z^{1+1} = z^2$$

$$\frac{\partial Z}{\partial x} = 2z\frac{\partial z}{\partial x}$$

$$\frac{\partial Z}{\partial x} = 2zp$$

$$zp = \frac{1}{2}\frac{\partial Z}{\partial x}$$

$$zp = \frac{1}{2}P.$$

$$\frac{\partial Z}{\partial y} = 2z\frac{\partial z}{\partial y}.$$

$$\frac{\partial Z}{\partial y} = 2zq$$

$$zq = \frac{1}{2}Q.$$

Sub. the values of pz and qz given equations. becomes...

$$\left(x+\frac{P}{2}\right)^2+\left(y+\frac{Q}{2}\right)^2=1.$$

$$\left(x+\frac{P}{2}\right)^2=1-\left(y+\frac{Q}{2}\right)^2 \quad \text{—— (2)}$$

This is of the form $f_1(x,P)=f_2(y,Q)$.

$$\left(x+\frac{P}{2}\right)^2=1-\left(y+\frac{Q}{2}\right)^2=a^2.$$

$$\left(x+\frac{P}{2}\right)^2=a \qquad 1-\left(y-\frac{Q}{2}\right)^2=a.$$

$$x+\frac{P}{2}=\sqrt{a} \qquad y-\frac{Q}{2}=\sqrt{1-a}.$$

$$x=\sqrt{a}+\frac{P}{2} \qquad Q=2\left[y-\sqrt{1-a}\right]$$

$$P=2\left(x-\sqrt{a}\right)$$

We have $dz=P\,dx+Q\,dy$

$$dz=2(x-\sqrt{a})\,dx+2\left[y-\sqrt{1-a}\right]dy$$

Integrating bothsides, weget

$$\int dz=2\int(x-\sqrt{a})\,dx+2\int\left[y-\sqrt{1-a}\right]dy+c$$

$$z=x^2-2\sqrt{a}\,x+y^2-\sqrt{1-a}.y+c.$$

$\therefore$ The Complete solution of (1) is

$$z^2=x^2-2\sqrt{a}\,x+y^2-\sqrt{1-a}\,y+c.$$

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Solve $p^2z^2\sin^2x + q^2z^2\cos^2y = 1$.

sol:- Given that $p^2z^2\sin^2x + q^2z^2\cos^2y = 1$. —— ①

It can be written as $(pz)^2\sin^2x + (qz)^2\cos^2y = 1$.

$$(pz)^2\sin^2x = 1-(qz)^2\cos^2y \quad \text{—— ②}$$

This is of the form. $f(x, pz^n) = g(y, qz^n)$ with $n=1$.

Put $Z = z^{n+1}$.

$Z = z^2$

$$\frac{\partial Z}{\partial x} = 2z\frac{\partial z}{\partial x} \qquad \frac{\partial Z}{\partial y} = 2z\frac{\partial z}{\partial y}$$

$$P = 2zp \qquad Q = 2z.q$$

$$zp = \frac{P}{2} \qquad zq = \frac{Q}{2}$$

Sub. the values of pz and qz in ②, we get

$$\left(\frac{P}{2}\right)^2\sin^2x = 1-\left(\frac{Q}{2}\right)^2\cos^2y$$

$$P^2\sin^2x = 4-Q^2\cos^2y.$$

This is of the form $f_1(x,P) = f_2(y,Q)$.

Let $P^2\sin^2x = 4-Q^2\cos y = a^2$.

Then $P^2\sin^2x = a^2$ and $4-Q^2\cos^2y = a^2$.

$P^2 = a^2\csc^2x$ and $Q^2\cos^2y = 4-a^2$

$P = a\csc x$. and $Q^2 = (4-a^2)\cos^2y$

$P = a\csc x$ and $Q = \sqrt{4-a^2}\cos y$.

We know that $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$

$$dZ = P\,dx + Q\,dy$$

$$dZ = a\csc x\,dx + \sqrt{4-a^2}\cos y\,dy$$

Solve $p^2 z^2 \sin^2 x + q^2 z^2 \cos^2 y = 1$.

sol:- Given that $p^2 z^2 \sin^2 x + q^2 z^2 \cos^2 y = 1$. —— ①

It can be written as $(pz)^2 \sin^2 x + (q^2 z)^2 \cos^2 y = 1$

$$(pz)^2 \sin^2 x = 1 - (qz)^2 \cos^2 y \quad \text{—— ②}$$

This is of the form. $f(x, pz^n) = g(y, qz^n)$ with $n = 1$.

Put $Z = z^{n+1}$

$Z = z^2$

$$\frac{\partial Z}{\partial x} = 2z \frac{\partial z}{\partial x} \qquad \frac{\partial Z}{\partial y} = 2z \frac{\partial z}{\partial y}$$

$$P = 2zp \qquad Q = 2z.q$$

$$zp = \frac{P}{2} \qquad zq = \frac{Q}{2}.$$

Sub. the values of pz and qz in ②, we get

$$\left(\frac{P}{2}\right)^2 \sin^2 x = 1 - \left(\frac{Q}{2}\right)^2 \cos^2 y$$

$$P^2 \sin^2 x = 4 - Q^2 \cos^2 y.$$

This is of the form $f_1(x, P) = f_2(y, Q)$.

Let $P^2 \sin^2 x = 4 - Q^2 \cos y = a^2$.

Then $P^2 \sin^2 x = a^2$ and $4 - Q^2 \cos^2 y = a^2$.

$P^2 = a^2 \csc^2 x$ and $Q^2 \cos^2 y = 4 - a^2$

$P = a \csc x$. and $Q^2 = (4 - a^2) \cos^2 y$

$P = a \csc x$ and $Q = \sqrt{4 - a^2} \cos y$.

We know that $dZ = \frac{\partial Z}{\partial x} dx + \frac{\partial Z}{\partial y} dy$

$$dZ = P\, dx + Q\, dy$$

$$dZ = a \csc x\, dx + \sqrt{4 - a^2} \cos y\, dy$$

Integrating bothsides, we get.

$$\int dz = a\int \csc x\, dx + \sqrt{1-a^2}\int \sec y\, dy + c.$$

$$z = a\log|\csc x - \cot x| + \sqrt{1-a^2}\log|\sec y + \tan y| + c.$$

$$z^2 = a\log|\csc x - \cot x| + \sqrt{1-a^2}\log|\sec y + \tan y| + c$$

Which is the complete sol. of the given P.D.E $\left[\because z = z^2\right]$

Solve $z(p^2-q^2) = x-y$.

Sol:- Given that $z(p^2-q^2) = x-y$. —— (1)

It can be written as $(\sqrt{z}\,p)^2 - (\sqrt{z}\,q)^2 = x-y$.

$$(z^{1/2}p)^2 - x = (z^{1/2}q)^2 - y \quad \text{—— (2)}.$$

This is of the form $f(x, pz^n) = g(y, qz^n)$ with $n=\frac{1}{2}$.

Put $Z = z^{n+1}$

$$Z = z^{\frac{1}{2}+1}$$

$$Z = z^{3/2}.$$

$$\frac{\partial Z}{\partial x} = \frac{3}{2}z^{1/2}\cdot\frac{\partial z}{\partial x}.$$

$$P = \frac{3}{2}z^{1/2}p.$$

$$z^{1/2}p = \frac{2}{3}P \quad \text{—— (3)}$$

$$\frac{\partial Z}{\partial y} = \frac{3}{2}z^{1/2}\frac{\partial z}{\partial y}.$$

$$Q = \frac{3}{2}z^{1/2}q$$

$$z^{1/2}q = \frac{2}{3}Q. \quad \text{—— (4)}$$

Sub. (3) and (4) in (2), we get

$$\left(\frac{2}{3}P\right)^2 - x = \left(\frac{2}{3}Q\right)^2 - y.$$

$$\frac{4}{9}P^2 - x = \frac{4}{9}Q^2 - y.$$

This is of the form $f_1(x, P) = f_2(y, Q)$.

Let $\frac{4}{9}P^2 - x = \frac{4}{9}Q^2 - y = a$.

Then $\frac{4}{9}P^2 - x = a$ and $\frac{4}{9}Q^2 - y = a$.

$\frac{4}{9}P^2 = a + x$ and $\frac{4}{9}Q^2 = a + y$

$P^2 = \frac{9}{4}(a+x)$ and $Q^2 = \frac{9}{4}(a+y)$

$P = \frac{3}{2}(a+x)^{1/2}$ and $Q = \frac{3}{2}(a+y)^{1/2}$.

We know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$.

$dz = P\,dx + Q\,dy$.

$dz = \frac{3}{2}(a+x)^{1/2}dx + \frac{3}{2}(a+y)^{1/2}dy$

Integrating bothsides, we get

$\int dz = \frac{3}{2}\int (a+x)^{1/2}dx + \frac{3}{2}\int (a+y)^{1/2}dy + c.$

$z = \frac{3}{2}\cdot\frac{2}{3}(a+x)^{3/2} + \frac{3}{2}\cdot\frac{2}{3}(a+y)^{3/2} + c.$

$z = (a+x)^{3/2} + (a+y)^{3/2} + c.$

$z^{3/2} = (a+x)^{3/2} + (a+y)^{3/2} + c \quad [\because z = z^{3/2}]$

Which is the complete sol. of given P.D.E.

Form the partial differential equation by eliminating arbitrary functions from $f(x^2+y^2-z^2,\ z^2-2xy)=0$.

Sol:- Given that $f(x^2+y^2-z^2,\ z^2-2xy)=0$

Which is in implicit form, where f is an arbitrary function.

Its explicit form can be written as $x^2+y^2+z^2=\phi(z^2-2xy)$ —(1).

Diff (1) w.r.t 'x' partially, we get.

$$2x+0-2z\frac{\partial z}{\partial x}=\phi'(z^2-2xy)\left[2z\frac{\partial z}{\partial x}-2y\right]$$

$$x-zp=\phi'(z^2-2xy)(zp-y) \quad \text{—(2)}$$

Diff (1) w.r.t 'y' partially, we get

$$0+2y-2z\frac{\partial z}{\partial y}=\phi'(z^2-2xy)\left[2z\frac{\partial z}{\partial y}-2x\right]$$

$$y-zq=\phi'(z^2-2xy)(zq-x) \quad \text{—(3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{x-zp}{y-zq}=\frac{\phi'(z^2-2xy)(zp-y)}{\phi'(z^2-2xy)(zq-x)}$$

$$(x-zp)(zq-x)=(y-zq)(zp-y)$$

$$pxz+xzq-x^2-z^2pq=yzp-y^2-z^2pq+yzq$$

$$pxz+xzq-x^2=yz(p+q)-y^2$$

$$xz(p+q)-x^2=yz(p+q)-y^2$$

$$xz(p+q)-yz(p+q)=x^2-y^2$$

$$z(p+q)(x-y)=(x+y)(x-y)$$

$$z(p+q)=x+y$$

Form the partial differential equation by eliminating the arbitrary function from $xy+yz+zx=f\left(\frac{z}{x+y}\right)$.

General Method of solving Equations of first order But of any Degree : Charpit's Methods

This method can be used to solve any first order partial differential equation. We prefer to use charpit's method only when it is not possible to use any of the special methods which we discussed earlier.

Working procedure of charpit's Method :

Step 1 :- Write the given equation as $f = 0$.

Step 2 :- Write charpit's auxiliary equations.

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}.$$

Step 3 :- Substitute the partial derivatives in the above auxiliary equations and simplify.

Step 4 :- choose two equations so that the resulting one on integration is a simple relation containing atleast one of p and q.

Step 5 :- solve the one obtained above with the given equation for p and q.

Step 6 :- Substitute these values of p and q in $dz = p\,dx + q\,dy$.

Step 7 :- Integrate the above equation to get the complete integral.

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Find the complete integral of $q = (z+px)^2$ using Charpit's Method.

Sol.- Given that $q = (z+px)^2$.

Let $f(x,y,z,p,q) = (z+px)^2 - q = 0$. —— ①

$f_x = 2(z+px)p$. $\quad f_y = 0 \quad f_z = 2(z+px)$

$f_p = 2(z+px)x \quad f_q = -1$.

Charpit's Auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}.$$

$f_x + pf_z = 4p(z+px)$

$f_y + qf_z = 2q(z+px)$

$-pf_p - qf_q = -2px(z+px) + q$

Sub. all these in above auxiliary equations, we get

$$\frac{dp}{4p(z+px)} = \frac{dq}{2q(z+px)} = \frac{dz}{-2px(z+px)+q} = \frac{dx}{-2x(z+px)} = \frac{dy}{1}$$

Taking 2nd and 4th members, we get

$$\frac{dq}{\cancel{2}q\cancel{(z+px)}} = \frac{dx}{-\cancel{2}x\cancel{(z+px)}}$$

Integrating bothsides, we get

$$\int \frac{dq}{q} = -\int \frac{dx}{x} + \log c.$$

$$\log|q| = -\log|x| + \log c.$$

$$\log|q| = \log\left|\frac{c}{x}\right|$$

$$q = \frac{c}{x} \quad . —— ②$$

Sub. (2) in (1), we get

$$(z+px)^2 - \frac{a}{x} = 0.$$

$$(z+px)^2 = \frac{a}{x}.$$

$$z+px = \sqrt{\frac{a}{x}}.$$

$$px = \sqrt{\frac{a}{x}} - z$$

$$p = \frac{\sqrt{a}}{x^{3/2}} - \frac{z}{x}.$$

We have $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$dz = p\,dx + q\,dy$$

$$dz = \sqrt{a}\, x^{-3/2} dx - \frac{z}{x} dx + \frac{a}{x} dy.$$

$$dz = \left(\frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}\right) dx + \frac{a}{x} dy$$

$$x\,dz = \sqrt{a}\, x^{-1/2} dx - z\,dx + a\,dy.$$

$$x\,dz + z\,dx = \sqrt{a}\, x^{-1/2} dx + a\,dy.$$

$$d(xz) = \sqrt{a}\, x^{-1/2} dx + a\,dy.$$

Integrating bothsides, we get

$$\int d(xz) = \sqrt{a} \int x^{-1/2} dx + a \int dy + c.$$

$$xz = 2\sqrt{a}\sqrt{x} + ay + c.$$

$$xz = 2\sqrt{ax} + ay + c.$$

Which is the complete sol. of given P.D.E.

Find a complete, singular and general integrals of $(p^2+q^2)y = qz$.

Sol:- Given that $(p^2+q^2)y = qz$

Let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ —— ①.

$f_x = 0 \qquad f_y = p^2+q^2 \qquad f_z = -q.$

$f_p = 2py \qquad f_q = 2qy - z$

Charpits Auxiliary equations are.

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$f_x + pf_z = -pq.$

$f_y + qf_z = p^2+q^2-q^2 = p^2$

$-pf_p - qf_q = -2p^2y - 2q^2y + qz$

Sub. all these in above auxiliary equations, we get.

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y-2q^2y+qz} = \frac{dx}{-2py} = \frac{dy}{z-2qy}$$

Taking the first two members, we get

$$\frac{dp}{-pq} = \frac{dq}{p^2} \implies \frac{dp}{-q} = \frac{dq}{p}$$

$$pdp + qdq = 0$$

Integrating both sides, we get.

$$\int pdp + \int qdq = c_1$$

$$\frac{p^2}{2} + \frac{q^2}{2} = c_1$$

$$p^2+q^2 = 2c_1$$

$$p^2+q^2 = a^2 \text{ where } a^2 = 2c_1. \quad \text{—— ②}$$

Sub. ② in ①, we get

$$a^2y - qz = 0$$

$$\implies q = \frac{a^2y}{z} \quad \text{—— ③}$$

Sub. ③ in ②, we get.

$$p^2 + \frac{a^4y^2}{z^2} = a^2$$

$$p^2 = a^2 - \frac{a^4y^2}{z^2}$$

$$p^2 = \frac{a}{z}\sqrt{z^2-a^2y^2}$$

We have $\quad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}dy.$$

$$z\,dz = a\sqrt{z^2-a^2y^2}\,dx + a^2y\,dy.$$

$$z\,dz - a^2y\,dy = a\sqrt{z^2-a^2y^2}\,dx$$

$$\frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx.$$

Integrating bothsides, we get

$$\frac{1}{2}\int \frac{2z\,dz - a^2 2y\,dy}{\sqrt{z^2-a^2y^2}} = a\int dx + c$$

$$\frac{1}{2}\cdot 2\sqrt{z^2-a^2y^2} = ax+c.$$

$$\sqrt{z^2-a^2y^2} = ax+c.$$

$$z^2-a^2y^2 = (ax+c)^2 \quad \text{—— ④}.$$

Which is a required complete integral. a, b being arbitrary constants.

Singular Integral :—

Diff ④ w.r.t a and b partially, we get

$$0 - 2ay^2 = 2(ax+c)x.$$

$$2ay^2 + 2x(ax+c) \quad \text{—— ⑤}$$

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$$0 = 2(ax+b) \quad \text{—— (6)}$$

Eliminating a and c between (4), (5) and (6), we get $z = 0$.

Which is clearly satisfies (1) and hence it is the singular integral.

General Integral :-

Replace c by $\phi(a)$ in (4), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2 \quad \text{—— (7)}.$$

Diff (7) w.r.t 'a' partially, we get.

$$-2ay^2 = 2[ax+\phi(a)][x+\phi'(a)] \quad \text{—— (8)}.$$

General integral is obtained by eliminating a from (7) and (8).