Computational Solution of Singularly Perturbed Two-Point Boundary Value Problems using Derivative Quadrature Method

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Abstract

This present research paper describes the application of Differential Quadrature Method (DQM) for getting the computational solution of singularly perturbed two point boundary value problems with varied condition in this method the concept based on the approximation of the derivatives of the unknown functions involved in the differential equations at the grid point of the solution domain. It is a significant discretization technique in solving initial and /or boundary value problems precisely using a considerably small number of mesh points. To test the applicability of the method we have solved several related model problems and presented the computational results. The computed results have been compared with the exact/approximate solution to exhibit the accuracy and efficiency of the developed technique.
Keywords- Differential Quadrature method; perturbation parameter, Singular perturbation; Ordinary differential equation, Boundary layer, Two –point boundary value problem, Deviating parameter

I. INTRODUCTION

The traditional finite difference method has an important limitation in practical applications, which is the requirement of a structured grid. The purpose of this paper is to introduce the Differential Quadrature method which is helpful on complex domains by replacing the derivatives as a weighing function. The application and analysis of the cubic spline is a continuous curve having some knots in the middle which is a method for all sort of model problems governed by the singular perturbation problems or convection-diffusion equation. The Stencil Mapping method is developed for complex domains.

II. DESCRIPTION OF THE DIFFERENTIAL QUADRATURE METHOD

The Differential Quadrature Method (DQM) was introduced by Bellman et al.[3] in the early 1970s and since then , the technique has been successfully employed for finding the solutions of many problems in applied and physical sciences. This process has been predicted by its proponents as a potential alternative to the conventional solution techniques such as the finite element and finite volume methods. The basic idea of the differential Quadrature method is that the derivative of a given function with respect to a space variable at a specific point is approximated as a weighted linear sum of the functional values at all discrete points in the domain of the variable.

In order to show the mathematical representation of the method, we select a one dimensional field variable. F(x) prescribed in a field domain $a=x_1 \leq x \leq x_n =b$. Let $f_i =f(x_i)$be the function values specified in a finite set of N discrete points x_i (I=1,2,3, …….N) of the field domain in which the end points x_1 and x_N are included. Next consider the value of the function derivative $\frac{d^r f}{dx^r}$ at some discrete points x_i, and let it is expressed as a linearly weighted some of the function values.

$$f^r(x_i) = \frac{d^r f(x_i)}{dx^r} = \sum_{j=1}^{N} A_{ij}^r \; f_j \quad (j=1,2,3,\ldots\ldots N) \tag{1}$$

Where A_{ij}^r are the weighting coefficients of the r^{th} order derivative of the function associated with point's x_i. The Quadrature rule is in equation (1) to obtain the derivative. Using equation (1) for various order derivatives, one may write a given differential equation at each point of its solution domain and get the Quadrature analog of the differential equation as a set of algebraic equations in terms of the N function values. These equations may be solved, in conjunction with the Quadrature method of the boundary conditions, to obtain the unknown function values provided that the weighting coefficients are known a priori.

For a detailed discussion on singular perturbation problems One can refer Kevorkian and cole[10], Bender and Orszag[4] and M. Stynes et. al[12].

In DQM , it is supposed that the solution of a one-dimensional differential equation is approximated by N-terms high degree polynomial.

$$f(x) = \sum_{k=1}^{n} c_k x^{k-1} \quad (2)$$

Where c_k is a constant.

The weighting coefficients may be determined by some appropriate functional approximations and the approximate functions are referred to as test functions. The primary requirements for the selection of the test functions are of differentiability and smoothness. Means the test function of the differential equations must be differentiable at least up to the n^{th} derivative (n is the highest derivative) and sufficiently smooth to be satisfied the condition of the differentiability . A convenient and most commonly used choice in one-dimensional problems is the Lagrangian interpolation shape functions$l_j(x)$, where

$$f(x) = \sum_{j=1}^{n} l_j(x) f_i \quad (3)$$

lj (x) are the monomials of the $(N-1)^{th}$ order polynomials . Observe that the number of test functions is equal to the number of the sampling points and for completeness, the number of the sampling points should at least be equal to one plus the order of the highest derivatives. Substituting $l_j(x)$ of equatin (3) in equation (1), the weighting coefficients can be easily obtained.

A. The Polynomial Test Function-Based Weighting Coefficients

The accuracy of differential Quadrature solution depends on the accuracy of the weighting coefficients. To obtain accurate weighting coefficients, Quan and Chang derived explicit formulae of the Lagrangian-interpolation –function based weighting coefficients for the first and second order derivatives. These formulae were obtained by taking the test function in the Lagrangian interpolation process as in equation (1) and (3). These explicit formulae's merit is that highly accurate weighting coefficients may be determined for any number of arbitrary spaced sampling points. In the weighting coefficients may be determined by some appropriate functional approximations and the approximate functions are referred to as test functions. The primary requirements for the selection of the test functions are of differentiability and smoothness. Means the test function of the differential equations must be differentiable at least up to the n^{th} derivative (n is the highest derivative) and sufficiently smooth to be satisfied the condition of the differentiability . A convenient and most commonly used choice in one-dimensional problems is the Lagrangian interpolation shape functions $l_j(x)$, where

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$$A_{i,j}^{(r)} = \frac{d^r}{dx^r} I_j(x_i) \quad (i,j = 1,2 ,3,\ldots\ldots\ldots N) \quad (4)$$

Where $I_j(x_i) = \frac{\emptyset(x)}{(x-x_j)\emptyset'_{x_j}}$; $\emptyset(x) = \prod_{m=1}^{N}(x - x_m)$

$\emptyset^1\ (x_j) = \frac{d\emptyset(x_j)}{dx} = \prod_{im=1,m\neq j}^{N}(x_j - x_m\ )$

X_i 's are the locations of the grid points. N is the number of smpling points .Here the equation (4) is valid as long as linearly independent polynomials are used as a trial functions and, thus thee values of the coefficients are affected only by the distribution of the grid points.

Note that the Lagrangian interpolation shape functions $l_i(x)$ has the following property

$$l_i(x) = \partial_{i,j} = \begin{cases} 1 & ,if\ \ i = j \\ 0 & , \text{if}\ \ i \neq j \end{cases} \quad (5)$$

Using equations (1), (3) and (4) based on Lagrangian interpolation shape functions , Quan and Chang[15] and Shu and rechards[18] obtained the following weighting coefficients.

$$A_{i,j}^{(1)} = \frac{dl_j(x_i)}{dx} = \frac{\emptyset'(x_i)}{\emptyset'(x_j)(x_i - x_j)} \quad (i,j = \ 1,2,3\ldots..\ N , i\neq j\ )$$

Where $l_j(x) = \frac{\emptyset(x)}{(x-x_j)(\emptyset'(x_j)}$; $\emptyset(x) = \prod_{m=1}^{N}(x - x_m)$;

$$A^{1}_{i,j} = \frac{dl_j(x_i)}{dx} = \frac{\emptyset^1(x_i)}{(x_i - x_j)(\emptyset^1 x_j)} , (i, j = 1,2,3 \dots \dots N;\ i \neq j)$$

$$A^{(r)}_{i,j} = \frac{d^r l_j(x_j)}{dx^r} = r\,(A^{r-1}_{i,j} A^{1}_{i,j} - \frac{A^{r-1}_{i,j}}{(x_i - x_j)} , i,j = 1,2\ 3 \dots\dots N ; i \neq j ; r \geq 2)$$

$$A^{(r)}_{i,j} = \frac{d^r l_i(x_j)}{dx^r} = - \sum_{j=1, i\neq j}^{n} A^{r}_{i,j} , (i = 1,2,3 \dots \dots . n; r \geq 1) \quad (6)$$

C. Selection of Sampling Points

A moderate and natural choice for the sampling points is that of the equally spaced points. But the Differential Quadrature solutions usually deliver more accurate results with unevenly spaced sampling points. A rational basis for the sampling points is provided by the zeros of the orthogonal polynomials. A well versed type of sampling point s in the DQM is the so called Gass-Chebyshev sampling points. For a domain specified by $a \leq x \leq b$ and discretized by a set of unequally spaced points then the coordinate of any point i can be evaluated by

$$x_i = a + \frac{1}{2}\left|1 - \cos\frac{i-1}{N-1}\pi\right| (b - a) \quad (7)$$

D. Application to Differential Equations

The basic key procedure in the DQM method is to approximate the derivatives in a differential equation by using equation (1) the associated equations, we can obtain simultaneous equations which can be solved by use of Gauss elimination method or other methods. Means DQM is composed the following steps.

The function to be determined is replaced by a group of function values at a group of selected sampling points. Gauss-Chebyshev-Labatto sampling points are strongly recommended for numerical stability.

1) Approximate derivatives in a differential equation by these N unknown function values.
2) Form a system of linear equations and solving the system of linear equation yields the desired unknowns.

The proper implementation of boundary condition is very important for the accurate numerical solution of the differential equation. Essential and natural boundary condition can be approximated by DQM. Using the technique in solving differential equation, the governing equations are actually satisfied at each sampling point of the domain, so one has one equation for each point, for each unknown. In the resulting system of the algebraic equation from the DQM, each boundary condition replaces the corresponding field equation. This method is simple and when there is one boundary condition at each boundary and when we have distributed the sampling points so that there is one point at each boundary.

E. Application to Singular Perturbation Problems

To show the applicability of DQM, we consider the singularly perturbed two point boundary value problems of the form

$$\varepsilon\, y''(x) + a(x)\, y'(x) + b(x)y(x) = g(x) ;\ p \leq x \leq q \quad (8)$$

With $d_i\, y(p) + d_2\, y'(p) = c_5$ (9)

And $d_3\, y(q) + d_4\, y'(q) = c_6$ (10)

Where ε is a small parameter $0 < \varepsilon \leq 1$. p,q, d_1, d_2, d_3, d_4 , c_5 ,c_6 are given constants . a(x), b(x) and g(x) are assumed to be sufficiently continuously differentiable functions in [p,q]. d_1, d_2, d_3, d_4 are non-zeros simultaneously .

For finding the solution of the governing equation (8) with the boundary conditions(Neumann (9) & (10) by Differential Quadrature method. One has to follow the method as illustrated below.

1) Discretize the interval [p, q] such that $p = x_1 < x_2 < \dots\dots\dots x_N = q$

Here N is the number of mesh points. Write $y_i = y(x_i)$ & $g_i = g(x_i)$

Apply the DQM to approximate the derivatives in the equations (8) , (9) & (10) that takes to the following discretized form of the equations .

$$\in \sum_{j=1}^{N} A^{(2)}_{i,j} Y_I + a_1 \sum_{j=1}^{N} A^{(1)}_{i,j} Y_I + b_I Y_I - g_I = 0 , \ I = 1,2,3 \dots\dots N. \quad (11)$$

With the boundary conditions

$$d_1 y_1 + d_2 \sum_{j=1}^{N} A^{(1)}_{i,j}\, y_j = c_5 \quad (12)$$

And

$$d_3 y_N + d_4 \sum_{j=1}^{N} A^{(1)}_{N,j}\, y_j = c_6 \quad (13)$$

use the equations (12)and (13) to solve for two unknown values y_1 &y_N

With the associated equations

$$Y_1 = \frac{1}{AXN}\left[C_5\left(d_3 + d_4 A^{1}_{N,N}\right) - c_6 d_2 A^{(1)}_{1,N} + \sum_{j=2}^{N-1} AXKY_1\right] \quad (14)$$

$$Y_N = \frac{1}{AXN}\left[C_6\left(d_1 + d_2 A^{1}_{1,1}\right) - c_5 d_4 A^{(1)}_{N,1} + \sum_{j=2}^{N-1} AXKNY_1\right] \quad (15)$$

$$\text{where } AXN = \left(d_1 + d_2 A^{(1)}_{1,1}\right)\left(d_3 + d_4 A^{(1)}_{N,N}\right) - d_2 d_4 A^{(1)}_{1,N} A^{(1)}_{N,1}$$

$$AXK_1 = d_2\ d_4 A^{(1)}_{1,N} A^{(1)}_{N,1} - \left(d_3 + d_4 A^{1}_{N,N}\right) d_2 A^{(1)}_{1,1}$$

$$\text{AXKN} = \quad d_2 d_4 A_{1,j}^{(1)} A_{N,1}^{(1)} - (d_1 + d_2 A_{1,1}^{(1)})\, d_4 A_{N,J}^{(1)}$$

2) Apply the equation (11) at all interior grid points x_i (i=1,2,3,……N-1) which gives to a system of N-2 equations with N unknowns.
3) Use the equation for y_1 and y_N from equation (14) and (15) in the obtained equations from step (2) to get another system of (N-2) equations with n-2 unknowns i.e. y_i,i=2,3,4,……..N-1.
4) Solve the above system of equations.
5) Apply the available values y_i,i=2,3,4,……..N-1 from step (3) in the equation (14) & (15) to get the approximate values of y at the boundary points $x=x_1$ and $x= x_N$.
6) Here we have applied Gauss-elimination (Back substitution way)
7) Partial pivoting and employed the C code to solve the obtained system of linear equations in the step (3) for the unknown values y_1 , y_2 , y_3,……..y_N .

III. NUMERICAL EXAMPLES

To show the applicability of the Differential Quadrature method we have employed this developed method to singular perturbation problems of linear or Non-linear behavior and computed the results for various values of N and ε . Some of the selected examples chosen because of they have been popularly discussed in the literature and exact solutions are available for comparison also measure the accuracy of the method.

Here important observation is that the Differential Quadrature method results are given at uniform grids x_i =ih with h=0.01 and K=100, which have interpolated from the use of Spline- interpolation polynomial. For the derivation of this polynomial , we have used the DQM results (x_i ,y_i), i=1,2…..N are the values of dependent variable at Non-Uniform grid points (Lagrangian) x_i obtained from equation (7) .

To show the accuracy and efficiency of the method we have also given the computational results (Using cubic spline interpolation polynomial) in terms of utmost Absolute error. (UAE) . for the examples (1.1) and (1.2) at a uniform grid k=100,1000 with h=0.01, 0.001 with the small parameter ε.Here the computations results can be given in terms of mean absolute error or the mean absolute percentage error or in terms of other types of error.

Example 1.1 Consider the following linear singular perturbation probem from Dorr et.al[6] & Andargie et.al[1].

$$\in y''(x) + y'(x) - y(x) = 0\,;\ \ 0 \le x \le 1 \text{ with } y'(0) = 0\,,\ y(1) + \in y'(1) = 1$$

In the above example a(x) =1 , b(x) =-1 and g(x) =0

$$\text{The exact solution is } y(x) = \frac{r_2\, e^{r_1 x} - r_1 e^{r_2 x}}{r_2(1+\in r_1)e^{r_1} - r_1(1+\in r_2)\, e^{r_2}}$$

$$\text{Here } r_1 = \frac{-1+\sqrt{1+4\in}}{2\in}\ ,\quad r_2 = \frac{-1-\sqrt{1+4\in}}{2\in}$$

Here the computational results are presented in Table 1.1 (a) in terms of Optimum Absolute error (OAE) for various value of N and∈. The table 1.1(b) show the comparison with exact and Andargie et.al [1] solution.

Table 1.1 (a) Maximum Absolute error in the Solution (Compiled by using cubic Spline interpolation poly.) for uniform points x_i =ih (i=0,1,2…..K) with h=0.01 and h=0.001 for the problem 1.1

Table 1.1(a):

ε ↓	N=16		N=32		N=64		N=80	
	K=100	K=1000	K=100	K=1000	K=100	K=1000	K=100	K=1000
10^{-1}	0.007594	0.007601	0.001786	0.001869	0.0004836	0.0004888	0.0003182	0.0003260
10^{-2}	0.009632	0.009632	0.2010	0.002012	0.0004852	0.0004888	0.0002765	0.0002879
10^{-3}	0.01650	0.01650	0.0027841	0.002817	0.0005096	0.0005121	0.0002962	0.0003052
10^{-4}	0.020134	0.02013	0.003660	0.004486	0.0007957	0.0008160	0.0004524	0.0004524
10-5	0.020532	0.02053	0.003797	0.004893	0.0009161	0.001155	0.0005478	0.0005585
10^{-6}	0.02057	0.02057	0.003811	0.004935	0.0009292	0.001194	0.0005609	0.0005984
10^{-9}	0.02057	0.02057	0.003813	0.004940	0.0009310	0.001199	0.0005633	0.0006002

Computational results for Example 1.1

Table 1.1(b):

Argument value(x)	Exact soln. y(x)	DQM Solution y(x) N=80 K=100, $\varepsilon = 10^{-4}$ O.A.E. 259041800E^{4}	Andargiae solution y(x), ε=10^{-4} ,h=0.01,δ=0.0008	DQM Solution y(x) N=35 K=100, $\varepsilon = 10^{-4}$ O.A.E .25904180E^{-3}
0.00	0.3679162	0.3680108	0.3691142	0.3679253
0.02	0.3751303	0.3751303	0.3757321	0.3753079
0.04	0.3829003	0.3829003	0.3833029	0.3828789
0.06	0.3905439	0.3905439	0.3910365	0.3906134
0.08	0.3985698	0.3986699	0.3989262	0.3985026
0.10	0.4063769	0.4063868	0.4069751	0.4065539
0.20	0.4494031	0.4494032	0.4497223	0.4493118
0.80	0.8186652	0.8101927	0.8188484	0.8186589

0.90	0.8914529	0.8854268	0.9154872	0.9258462
1.00	0.9999000	0.9998693	0.9999001	0.9998859

Example 1.12 Consider the singular perturbation problem from Dorr et.al[6] with a=1 and n=1 $-\epsilon\, u''(x) + u'(x) + u(x) = 0\,;\, u(0) - u'(0) = 1\,, u'(1) = 0\,, 0 \le x \le 1$

The computational results are presented in table 1.2(a) , in terms of Maximum Absolute error for various values of N and ε. The Table 1.2(b) compares with the exact and Andargie et.al[1]

For this example 1.12 a(x) =-1, b(x) =-1 and g(x) =0

The exact solution is given by $u(x) = \frac{r_1 e^{r_2 x} - r_2 e^{r_1(x-1)+r_2}}{r_1(1-\epsilon r_2) - r_2(1-\epsilon r_1)\, e^{(r_2-r_1)}}$ where

$$r_1 = \frac{1+\sqrt{1+4\epsilon}}{2\epsilon}\ ,\quad r_2 = \frac{1-\sqrt{1+4\epsilon}}{2\epsilon}$$

The computational results are presented in Table 2.2(a) and 2.2(b) for different values of N and ε. This example exhibits the right end boundary layer behavior.

Table 1.2(a): Computational results for the example1.12

Argument value(x)	Exact soln. y(x)	DQM Solution y(x) ; N=46 K=100, $\varepsilon = 10^{-4}$ O.A.E. 0.1854896E^{-3}	Andargie Solution y(x), ε=10^{-4} ,h=0.01,δ=0.0008	DQM Solution y(x) N=98 K=100, $\varepsilon = 10^{-4}$ O.A.E .2855062E^{-4}
0.00	0.9999000	0.9999282	0.9999009	0.9999105
0.20	0.8186653	0.8186924	0.8188949	0.8188948
0.40	0.6702799	0.6702996	0.6706179	0.6706179
0.60	0.5487997	0.5488139	0.5491892	0.5491892
0.80	0.4493200	0.4493309	0.4497477	0.4497477
0.90	0.4065656	0.4065794	0.4069981	0.4069981
0.92	0.3985159	0.3985258	0.3989487	0.3989487
0.94	0.3906255	0.3906269	0.3910585	0.3910598
0.96	0.3828916	0.3828987	0.3833245	0.3833246
0.98	0.3753105	0.3752099	0.3757532	0.3757532
1.00	0.3679162	0.3679299	0.3691349	0.3691349

Table 1.2(b):

Argument value(x)	Exact soln. y(x)	DQM Solution y(x) ; N=56 K=100, $\varepsilon = 10^{-5}$ O.A.E. 0.1195669E^{-3}	DQM Solution y(x) N=88 K=100, $\varepsilon = 10^{-5}$ O.A.E 0.4959109E^{-4}
0.00	0.9999000	0.9999935	0.9999933
0.20	0.8186653	0.8187791	0.8187711
0.40	0.6702799	0.6703058	0.6703593
0.60	0.5487997	0.5489030	0.5489054
0.80	0.4493200	0.4494078	0.4493231
0.90	0.4065656	0.4065762	0.4065796
0.92	0.3985159	0.4025403	0.3985339
0.94	0.3906255	0.3906401	0.3906351
0.96	0.3828916	0.3715862	0.3828842
0.98	0.3753105	0.3752778	0.3753292
1.00	0.3679162	0.3678831	0.3679164

Example 1.3: Consider the following homogeneous singular perturbation problem

$$\varepsilon y''(x) + y'(x) = 1 + 2x, \quad 0 \le x \le 1$$

$$with\ \ y(0) = 0\ and\ y(1) = 1$$

$$y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{\frac{-x}{\varepsilon}})}{(1 - e^{\frac{-x}{\varepsilon}})}$$

We have solved the above problem with $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively. The approximate solutions obtained by the proposed method described in chapter-2 are compared with the exact solution in tables 3.2 a and 3.2 b for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-4}$ respectively. From the results we can conclude that the approximation is in good agreement with the exact solution.

Table 1.3(a):

Argument value(x)	DQM Approximate Solution y(x) ; N=56 K=100, $\varepsilon = 10^{-3}$, δ=0.009 M. A.E 0.1195669E^{-3}	Exact Solution
0.00000	-0.00000002	0.00000000
0.00200	-0.85699360	-0.86093540
0.55000	-0.14659230	-0.14660000
0.60000	-0.03916838	-0.03919995
0.70000	0.19060910	0.19060000

0.80000	0.44043510	0.44040000
0.90000	0.71021060	0.71020000
1.00000	1.00000000	1.00000000

Example For the above problem 1.3 with $\varepsilon = 10^{-4}$, h= 0.0015 &δ = 0.009 With DQM method the results are as given below.

Table 1.3(b):

Argument value(x)	DQM Approximate Solution y(x) ; N=56 K=80, $\varepsilon = 10^{-4}$,	Exact solution
0.00000	0.00000000	0.00000000
0.00020	-0.86029370	-0.86429180
0.50000	-0.24994970	-0.24990000
0.550000	-0.14742380	-0.14741000
0.60000	-0.03996952	-0.03991995
0.65000	0.07255605	0.07256994
0.70000	0.19001050	0.19006000
0.75000	0.31253600	0.31255000
0.80000	0.43999070	0.44004000
0.85000	0.57251580	0.57253010
0.90000	0.70997110	0.71001990
0.95000	0.85249320	0.85251000
1.00000	1.00000000	1.00000000

Example 1.4: Consider the following homogeneous Singular value perturbation problem from Kevorkian and Cole [10] with α =0:

$$\varepsilon\, y''(x) + y'(x) = 0,\ \ 0 \le x \le 1 \text{ with } y(0) = 0 \text{ and } y(1) = 1$$

The exact solution is given by

$$y(x) = \frac{(1 - \exp(-x/\varepsilon))}{(1 - \exp(-1/\varepsilon))}$$

The computational results are presented in Table 1(a) and (b) for ε = 10^{-3}, 10^{-4} respectively.

Table 1.4(a):

X	DQM Approximate Solution y(x) ; N=56 K=80, $\varepsilon = 10^{-3}$			Exact solution
ε=0.001,h=0.01	δ=0.008	δ=0.009	δ=0.007	
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	0.9876486	0.9899944	0.9917358	1.0000000
0.04	0.9998419	0.9998944	0.9999319	1.0000000
0.06	0.9999925	0.9999934	0.9999995	1.0000000
0.40	0.9999964	0.9999964	1.0000000	1.0000000
0.60	0.9999976	0.9999976	1.0000000	1.0000000
0.80	0.9999988	0.9999988	1.0000000	1.0000000
1.00	1.00000000	1.00000000	1.0000000	1.0000000

(b) F or the above problem again selecting the values ε= 10^{-4} and h= 0.01 the computed results by using DQM method as follows.

Table 1.4(b):

Argument value(x)	DQM Approximate Solution y(x) ; N=56 K=80, $\varepsilon = 10^{-3}$			Exact solution
h=0.02	δ=0.008	δ=0.009	δ=0.007	
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.02	0.9998016	0.9998477	0.9998792	1.0000000
0.04	0.9999999	1.0000000	1.0000000	1.0000000
0.60	1.0000000	1.0000000	1.0000000	1.0000000
0.80	1.0000000	1.0000000	1.0000000	1.0000000
1.00	1.0000000	1.0000000	1.0000000	1.0000000

IV. RESULT ANALYSIS AND CONCLUSIONS

In the presented paper the Differential Quadrature method (DQM) applied to solve linear singular perturbation problems with boundary layer (Left, right) The major applications presented here revealed that the method has the erudite efficiency of solving general singularly perturbed two point boundary value problems with Dirichlet's boundary conditions, Neumann's boundary conditions & mixed boundary conditions, and also getting approximate solutions with minimal computations. It can be viewed from the results that the methods approximated the exact solution or asymptotic or approximate solution very much with small number of sampling points. This exhibits the accuracy and efficiency of the method. Here we have given few values although the solutions can be compiled at desired number of equally spaced points.

It has been observed that the computed approximate solution matches with the exact solution very fair indicates the efficiency of the DQM method. Also it has been observed that increase in the number of mesh points gives rise to an increase in the accuracy of the solution, as similar with the most numerical methods. However a small number of grid points in DWM give highly accurate results with the use of non-uniform mesh points. This method gives a supplementary technique to the conventional ways of solving singular perturbation problems.

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