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PRIME ORDERED K-BI-IDEALS IN ORDERED TERNARY SEMIRINGS

Mantha Srikanth[1, 2] and G. Shobhalatha[3]

[1,3]Department of Mathematics, Sri Krishnadevaraya University, Ananthapuramu, Andhra Pradesh,India-515003

[2]Department of Mathematics, Malla Reddy Engineering College (Autonomous), Hyderabad,Telangana, India-500100

Abstract:

The notion of Bi-ideals, K-bi-ideals, prime K-bi-ideals of an ordered ternary semirings are introduced and several characterizations are obtained through the ordered relations with some propertieson them.

Keywords: Ordered ternary semiring, Ordered K-bi-ideal, Prime K-bi-ideal, K-Closure, stronglyirreducible, fully ordered K-bi-ideal in ternary semirings.

1. Introduction

Algebraic structures play a very outstanding role in mathematics with wide ranging applications in multivarious disciplines such as theoretical physics, computer sciences, control engineering, coding theory, graph theory, probability theory, optimization theory, automata theory etc.

The notion of ternary algebraic system was introduced by Lehmer [7] in 1932. He investigatedcertain ternary algebraic systems called triplexes. Dutta and kar[6] introduced a notion of ternary semirings which is a generalization of semirings and they studied some properties of ternary semirings([6],[8],[10]). K Arulmozhi was introduced new types of ideals in semirings and ternary semirings [4]. Jones and Johnson [1] introduced Prime ideals and congruences in ternary semirings. Wang and Li [2] introduced Prime ideals and irreducible elements in ternary semirings. Mursaleen [3] introduced prime ideals and maximal ideals in ternary semirings. Han, Kim and Neggers [5] investigatedproperties orders in a semiring. Prime ordered k-bi-idelas in ordered ternary semirings was introducedby Permpoon Senarat, Bundit Pibaljommee[11].

In this paper, we define notion of prime ordered K-bi-ideal, a strongly prime ordered K-bi-ideal,an irreducible ordered K-bi-ideal and a strongly irreducible ordered K-bi-ideal of an ordered ternary semiring. We introduce the new concept of an ordered K-bi-idempotent ternary semirings and characterize it using prime, strongly prime, irreducible and strongly irreducible ordered K-bi-ideals. And we study the various properties of these ideals in an ordered ternary semirings. These structures we use in Boolean algebra, automata theory etc.

2. Preliminaries

This section includes a few definitions and observations that will help us interpret our main results.

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Definition:2.1. A semiring is a triplet (T, +, ·) consisting of a nonempty set T and two operations +(addition) and · (multiplication) such that

i) (T, +) is a commutative semigroup
ii) (T, ·) is a semigroup
iii) $x.(y + z) = x.y + x.z$ and $(x + y).z = x.z + y.z$ for all $x, y, z \in T$.

Definition:2.2. A semiring (T, +, ·) is called a commutative if (T, ·) is a commutative semigroup. An element 0 ∈ T is called a zero element if $x + 0 = 0 + x = x$ and $x. 0 = 0 = 0. x$.

Definition:2.3. A nonempty subset B of a semiring (T, +, ·) is called a left (right) ideal of T if $x + y \in B$ for all x, y ∈ B and TB⊆B (BT⊆B). We call B an ideal of T if it is both a left and a right ideal of T.

Definition:2.4. A subsemiring 'I' of Semiring T is called a bi-ideal of T if $ITI \subseteq I$.

Definition:2.5. A non-empty set T together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition is said to be a ternary semiring if (T, +) is a commutative semigroup and ternary multiplication satisfies the following:

(i) $(xyz)uv = x(yzu)v = xy(zuv)$,
(ii) $(x + y)zu = xzu + yzu$,
(iii) $x(y + z)u = xyu + xzu$,
(iv) $xy(z + u) = xyz + xyu$, *for all* $x, y, z, u, v \in T$.

Definition:2.6. An ordered ternary seniring $(T, +, . , \leq)$ is such that $(T, +, .)$ is a ternary semiring and $(T, \leq)$ is a partially ordered set. And the relation ≤ is compatible to the operations '**+**' and '**.**'. i.e., if $a \leq b$, then $a + x \leq b + x$, $x + a \leq x + b$, $ax \leq bx$ *and* $xa \leq bx$, for all $a, b, x \in T$.

Definition:2.7. A ternary subsemiring 'I' of an ordered ternary semiring 'T' is called a Bi-ideal of T if $ITITI \subseteq I$ and $b \in I, a \in T, a \leq b \Longrightarrow a \in I$.

Definition:2.8. An ordered ternary semiring T is called I -Simple if it has no non-zero proper bi-ideals. A proper Bi-ideal 'I' of 'T' is called prime if $PQR \subseteq I \Longrightarrow P \subseteq I$ *or* $Q \subseteq I$ *or* $R \subseteq I$. For any three bi- ideals of P, Q, R of T.

Let $(T, +, . , \leq)$ be an ordered ternary semiring. For nonempty subsets P, Q and R of 'T' and a ∈T, we define

(i) $[\mathrm{P}) = \{\mathrm{x} \in \mathrm{T} / \mathrm{a} \leq \mathrm{x} \text{ for some } \mathrm{a} \in \mathrm{P}\}$
(ii) $[\mathrm{PQR}) = \{\mathrm{xyz} \in \mathrm{T} / \mathrm{x} \in \mathrm{P}, \mathrm{y} \in \mathrm{Q} \text{ and } \mathrm{z} \in \mathrm{R}\}$,
(iii) $P = \{\sum_{i\in \mathrm{J}} \mathrm{a_i} \in \mathrm{T} / \mathrm{a_i} \in \mathrm{P}, \text{and J is a finite subset of } \mathbb{N}\}$,
(iv) $PQR = \left\{\sum_{i\in \mathrm{J}} \mathrm{a_i}\, \mathrm{b_i} \mathrm{c_i} \in \mathrm{T} / \mathrm{a_i} \in \mathrm{P}, \mathrm{a_i} \in \mathrm{P}, \mathrm{a_i} \in \mathrm{P} \text{and J is a finite subset of } \mathbb{N}\right\}$
(v) $\mathbb{N}\mathrm{a} = \{\mathrm{na} \in \mathrm{T} / \mathrm{n} \in \mathbb{N}\}$.

Instead of writing an ordered ternary semiring $(T, +, . , \leq)$ we simply denote T as an ordered

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ternary semiring.
A left (right) ideal P of an ordered ternary semiring T is called a left (right) ordered ideal of T if for any
$a \leq x$ for some $a \in P \Longrightarrow x \in P$. We call P an ordered ideal if it is both a left and a right ordered ideal of T.
A left (right) ordered ideal of P of an ordered ternary semiring T is called a left (right) ordered k-ideal of T if $a + x = b$ for some $a, b \in P \Longrightarrow x \in P$. We call P an ordered k-ideal of T if it is both a left and a right ordered k-ideal of T.
The k-closure of a nonempty subset P of an ordered ternarysemiring T is defined by
$\underline{P} = \{x \in T / \exists\ a, b \in P, b \leq x + a\}$.
Now, we recall the modified results concerning to the k-closure given in [8].
Lemma 2.1. Let T be an ordered ternarysemiring and A, B be nonempty subsets of T.

(i) $[\overline{\mathrm{P}}) \subseteq \overline{[\mathrm{P})}$.
(ii) If $\mathrm{P} \subseteq \mathrm{Q} \subseteq \mathrm{R}$, then $\overline{\mathrm{P}} \subseteq \overline{\mathrm{Q}} \subseteq \overline{\mathrm{R}}$.
(iii) $\overline{[\mathrm{P})}\mathrm{QR} \subseteq \overline{[\mathrm{PQR})}$, $P\overline{[\mathrm{Q})}\mathrm{R} \subseteq \overline{[\mathrm{PQR})}$ and $PQ\overline{[\mathrm{R})} \subseteq \overline{[\mathrm{PQR})}$.

Lemma 2.2. Let P be a nonempty subset of an ordered ternary semiring T. If P is closed under addition, then [P) and $\overline{[\mathrm{P})}$ are also closed.
Lemma 2.3. Let T be an ordered semiring and P, Q and R are nonempty subsets of T with $\mathrm{P} + \mathrm{P} + \mathrm{P} \subseteq \mathrm{P}$and $\mathrm{Q} + \mathrm{Q} + \mathrm{Q} \subseteq \mathrm{Q}$. Then

(i) $\mathrm{P} \subseteq [\mathrm{P}) \subseteq \overline{\mathrm{P}} \subseteq \overline{[\mathrm{P})}$;
(ii) $\overline{[\mathrm{P})} = \overline{\overline{[\mathrm{P})}}$;
(iii) $\mathrm{P} + \mathrm{Q} + R \subseteq \overline{\mathrm{P}} + \overline{\mathrm{Q}} + \overline{\mathrm{R}} \subseteq \overline{\mathrm{P} + \mathrm{Q} + \mathrm{R}}$;
(iv) $\overline{[\mathrm{P})} + \overline{[\mathrm{Q})} + \overline{[\mathrm{R})} \subseteq \overline{\overline{[\mathrm{P})} + \overline{[\mathrm{Q})} + \overline{[\mathrm{R})}} \subseteq \overline{[\mathrm{P} + \mathrm{Q} + \mathrm{R})}$;
(v) $\overline{\mathrm{P}}\,\overline{\mathrm{Q}}\overline{\mathrm{R}} \subseteq \overline{[\mathrm{P})[\mathrm{Q})[\mathrm{R})} \subseteq \overline{[\sum \mathrm{PQR})}$;
(vi) $\mathrm{P}(\sum \mathrm{QR}) \subseteq (\sum \mathrm{PQR})$,$P(\sum \mathrm{Q})\mathrm{R} \subseteq (\sum \mathrm{PQR})$ and $PQ(\sum \mathrm{R}) \subseteq (\sum \mathrm{PQR})$.

Lemma 2.4. Let T be an ordered ternary semiring and P be a nonempty subset of T with $\mathrm{P} + \mathrm{P} \subseteq \mathrm{P}$.
Then $\overline{[\overline{[\mathrm{P})})} = \overline{[\mathrm{P})}$.
For a nonempty subset P of an ordered ternary semiring T, we denote by
$\mathrm{L_k(P)}, \mathrm{M_k(P)}, \mathrm{N_k(P)}$
and $\mathrm{Q_k(P)}$the smallest left ordered k-ideal, the smallest lateral ordered k-ideal, the smallest right ordered k-ideal and the smallestordered k-ideal of T containing P, respectively. For any x ∈ T we denote
$\mathrm{L_k(x)} = \mathrm{L_k}(\{x\}), \mathrm{M_k(x)} = \mathrm{M_k}(\{\mathrm{x}\}), \mathrm{N_k(x)} = \mathrm{N_k}(\{x\})$. and $\mathrm{Q_k(x)} = \mathrm{Q_k}(\{x\})$.

Theorem 2.1. Let T be an ordered ternary semiring and x ∈ T, then

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(i) $L_k(P) = \overline{[\Sigma P + \Sigma TTP)}$;
(ii) $M_k(P) = \overline{[\Sigma P + \Sigma TPT)}$;
(iii) $N_k(P) = \overline{[\Sigma P + \Sigma PTT)}$;
(iv) $Q_k(x) = \overline{[\Sigma P + \Sigma TTP + \Sigma TPT + \Sigma PTT + \Sigma TPTPT)}$.

Corollary 2.1. Let T be an ordered semiring and x ∈ T, then

(i) $L_k(x) = \overline{[\mathbb{N}x + TTx)}$;
(ii) $M_k(x) = \overline{[\mathbb{N}x + TxT)}$;
(iii) $N_k(x) = \overline{[\mathbb{N}x + xTT)}$;
(iv) $Q_k(x) = \overline{[\mathbb{N}x + TTx + TxT + xTT + \Sigma TxTxT)}$.

1. Ordered K-bi-ideal of an Ordered Ternary Semiring

Definition:3.1. An ordered subsemiring 'I' of an ordered ternary semiring 'T' is said to be an ordered K-bi- ideal of T if

i) ITITI ⊆ I
ii) If $x \in T, a + x = b$ for some $a, b \in$ Ithen $x \in$ T
iii) If $x \in T$, $b \leq$ x for some $b \in$ I then $x \in$ T

Theorem 3.1. Let 'I' be a bi-ideal of an ordered ternary semiring T. Then the following statements are equivalent.

(i) Iis an ordered K-bi-ideal of S.
(ii) If $b \leq$ a + xfor some $a, b \in$ I, then x ∈I.
(iii) $\bar{I} = I$.

Proof. Assume that 'I' be a bi-ideal of an ordered ternary semiring T.

(i) ⇒ (ii): Let I be an ordered K-bi-ideal of an ordered ternary semiring T.

If $b \leq$ a + x for some $a, b \in$ I and $x \in$ T. Then a + x ∈ I. It follows that there exists p∈I such that

a + x = p. By assumption, x ∈T.

(ii) ⇒ (iii): Let $x \in \bar{I}$. Then ∃ $a, b \in$ I such that $b \leq$ a + x.

By assumption, we have $x \in$ I.

Therefore $x \in \bar{I} \Rightarrow x \in$ I.Thus, $\bar{I} = I$.

(iii) ⇒ (i): Suppose that $\bar{I} = I$.

Let $x \in$ T such that $a + x = b$ for some $a, b \in$ I. Then $x \in \bar{I}$.

By assumption, we have $x \in$ I.

By Lemma 2.3(i), [I) ⊆ $\bar{I}$ = I. Therefore 'I' is an ordered K-bi-ideal of an ordered ternary semiring T.

Theorem 3.2. Let I be a bi-ideal of an ordered ternary semiring T. Then $\overline{[I)}$ is the smallest ordered K-bi-ideal of an ordered ternary semiring T containing 'I'.

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Proof. It is clear that $I \subseteq \overline{[I)}$.
By Lemma 2.2, $\overline{[I)}$ is closed under addition.
By Lemma 2.1(iii) and Lemma 2.4, $\overline{\overline{[I)}\,\overline{[I)}\,\overline{[I)}} \subseteq \overline{[\overline{[I)}\,\overline{[I)})\overline{[I)})}$
$\subseteq \overline{[\overline{[I\,I\,I)})}$
$\subseteq \overline{[\overline{[I)})}$
$=\overline{[I)}$.
By Lemma 2.1(iii), $\overline{[I)}\ T\ \overline{[I)}\ T\ \overline{[I)} \subseteq \overline{[\Sigma ITITI)}$
$\subseteq \overline{[I)}$.
Thus $\overline{[I)}$ is a bi-ideal of T.
By Lemma 2.3(ii), $\overline{[\overline{[I)})} = \overline{[I)}$.
By Theorem 3.1, $\overline{[I)}$ is an ordered K-bi-ideal of ordered ternary semiring T.
Let P be an ordered K-bi-ideal of T containing I. Then $[I) \subseteq [P) = P$ and $\overline{[I)} \subseteq \overline{P} = P$.
Then $\overline{[I)}$ is the smallest ordered K-bi-ideal of an ordered ternary semiring T containing I.

2. Prime Ordered K-bi ideal of an Ordered Ternary Semiring

Theorem 4.1. Intersection of a family of ordered K-bi-ideals of an ordered ternary semiring T is an ordered K-bi-ideal of T.
Definition 4.1. An ordered K-bi-ideal I of an ordered ternary semiring T is called a semiprime ordered K-bi-ideal if $\overline{[\Sigma PPP)} = \overline{[\Sigma P^3)} \subseteq$ Iimplies P⊆ I for any ordered K-bi-ideal ‘I’ of T.
Definition 4.2. An ordered K-bi-ideal ‘I’ of ordered ternary semiring T is called a prime ordered K-bi-ideal if $\overline{[\Sigma PQR)} \subseteq I$ implies $P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$ for any ordered K-bi-ideals P, Q and R of T.
Definition 4.3. An ordered K-bi-ideal ‘I’ of an ordered ternary semiring T is called a strongly prime ordered K-bi-ideal if $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} \subseteq I$ implies $P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$ for any ordered K-bi-ideals P, Q and R of T.

Obviously, every strongly prime ordered k-bi-ideal of T is a prime ordered K-bi-ideal and every prime ordered K-bi-ideal of T is a semiprime ordered K-bi-ideal.
The following example shows that every prime ordered K-bi-ideal need not to be a strongly prime ordered K-bi-ideal.
Example 4.1. Let $T = \{x, y, z\}$. We define operations + and · on T as the follows.

$+$	x	y	z
x	x	y	z
z	z	y	z
z	z	z	z

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and

.	x	y	z
x	x	x	x
y	x	z	z
z	x	z	z

We defined a partially ordered relation $\leq$ on T by $\leq := \{(x,x),(y,y),(z,z),(x,y)\}$.
We can show that $(T,+,.,\leq)$ is an ordered ternary semiring and $\{x\},\{x,y\},\{x,z\}$ *and* T are all ordered K-bi-ideals of T. Now, we have $\{x\}$ is prime but not strongly prime, since $\overline{(\Sigma\{x,y\}\{x,z\})} \cap \overline{(\Sigma\{x,z\}\{x,y\})} = \{x\}$ but $\{x,y\} \nsubseteq \{x\}$ and $\{x,z\} \nsubseteq \{x\}$.

Example 4.2. Let $T = \{x,y,z,u,v,w\}$. We define operations + and $\cdot$ on T as the follows.

$+$	x	y	z	u	v	w
x	x	y	z	u	v	w
y	y	y	z	u	v	w
z	z	z	z	v	v	w
u	u	u	v	u	v	w
v	v	v	v	v	v	w
w	w	w	w	w	w	w

and

.	x	y	z	u	v	w
x	x	x	x	x	x	x
y	x	x	x	y	y	z
z	x	y	z	y	z	z
u	x	x	x	u	u	w
v	x	y	z	u	v	w
w	x	u	w	u	w	w

We defined a partially ordered relation $\leq$ on T by
$\leq$
$:= \{(x,x),(y,y),(z,z),(u,u),(v,v,),(w,w),(x,y)(x,z),(x,u),(x,v),(y,z),(y,u),(y,v),$

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$(z,u),(u.v)\}$.

The sets $A=\{x\},\ B=\{x,y\}, C=\{x,y,z\}, D=\{x,y,u\}$ and T are all ordered K-bi-ideals of T.

We find that C, D and Tare strongly prime ordered K-bi-ideals, B is a semiprime ordered K-bi-ideal but not prime and A is not a semiprime ordered K-bi-ideal.

Definition 4.4. An ordered K-bi-ideal I of an ordered ternary semiring T is called an irreducible ordered K-bi-ideal if for any ordered K-bi-ideal P, Q and R of T, $P \cap Q \cap R = I$ implies $P = I$ or $Q = I$ or $R = I$.

Definition 4.5. An ordered K-bi-ideal I of an ordered ternary semiring T is called an strongly irreducible ordered K-bi-ideal if for any ordered K-bi-ideal P, Q and R of T, $P \cap Q \cap R \subseteq I$implies $P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$.

It is clear that every strongly irreducible ordered K-bi-ideal of an ordered ternary semiring T is an irreducible ordered K-bi-ideal of T.

Theorem 4.2. If intersection of a family of prime ordered K-bi-ideals (or semiprime ordered K-bi-ideals) of an ordered ternary semiring T is not empty, then itis a prime ordered K-bi-ideal (or semiprime ordered K-bi-ideal) of ordered ternary semiring T.

Theorem 4.3. If 'I' is a strongly irreducible and semiprime ordered K-bi-ideal of an ordered ternary semiring T, then 'I' is a strongly prime ordered k-bi-ideal of T.

Proof. Assume 'I' be a strongly irreducible and semiprime ordered K-bi-ideal of an ordered ternary semiring T.

Let P, Q and R be any threeordered K-bi-ideal of T sch that

$\overline{(\Sigma PQR]} \cap \overline{(\Sigma QRP]} \cap \overline{(\Sigma RPQ]} \subseteq I$

Since $\overline{(\Sigma(P \cap Q \cap R)^3]} \subseteq \overline{(\Sigma PQR]}$,

$\overline{(\Sigma(P \cap Q \cap R)^3]} \subseteq \overline{(\Sigma QRP]}$ and

$\overline{(\Sigma(P \cap Q \cap R)^3]} \subseteq \overline{(\Sigma RPQ]}$

We have $\overline{(\Sigma(P \cap Q \cap R)^3]} \subseteq \overline{(\Sigma PQR]} \cap \overline{(\Sigma QRP]} \cap \overline{(\Sigma RPQ]}$.

Since $P \cap Q \cap R$ is an ordered K-bi-ideal and 'I' is a semiprime ordered K-bi-ideal, $P \cap Q \cap R \subseteq I$.

Since Iis a strongly irreducible ordered K-bi-ideal, $P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$.

Thus, 'I' is a strongly prime ordered k-bi-ideal of an ordered ternary semiring T.

Theorem 4.4. If 'I' is an ordered K-bi-ideal of an ordered ternary semiring T and $x \in T$ such that $x \notin I$, then $\exists$ an irreducible ordered K-bi-ideal M of T such that $I \subseteq M$ and $x \notin M$.

Proof. Let $\mathcal{K}$ be the set of all ordered K-bi-ideals of T containing 'I' but not containing x. Then $\mathcal{K}$ is a nonempty set, since $I \in \mathcal{K}$. Clearly, $\mathcal{K}$ is a partially ordered set under the inclusion of sets.

Let $\mathcal{N}$ be a chain subset of $\mathcal{K}$. Then $\cup \mathcal{N} \in \mathcal{K}$. By Zorn's Lemma, there exists a max.

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element in $\mathcal{K}$.

Let M be a max. element in $\mathcal{K}$.

Let P, Q and R be any three ordered K-bi-ideals of T $\ni P \cap Q \cap R = M$.

Suppose that $M \subset P$ and $M \subset Q$ and $M \subset R$.

Since M is a max. element in $\mathcal{K}$, we have $x \in P$ and $x \in Q$ and $x \in R$. Then $x \in P \cap Q \cap R = M$, which is a contradiction.

Thus, $R = M$ or $Q = M$ or $P = M$.

Therefore, M is an irreducible ordered k-bi-ideal of an ordered ternary semiring T.

Theorem 4.5. A prime ordered K-bi-ideal 'I' of an ordered ternary semiring T is a prime one sided ordered K-ideal of T.

Proof. Let 'I' be a prime ordered K-bi-ideal of T.

Suppose that 'I'is not a one-sided ordered K-ideal of T.

It follows that $\overline{[ITT)} \nsubseteq I$ and $\overline{[TIT)} \nsubseteq I$ and $\overline{[TTI)} \nsubseteq I$.

Then $\overline{[\sum ITT)} \nsubseteq I$ and $\overline{[\sum TIT)} \nsubseteq I$ and $\overline{[\sum TTI)} \nsubseteq I$.

Since 'I'is a prime ordered K-bi-ideal, $\overline{[\sum \overline{[\sum ITT)}\overline{[\sum TIT)}\overline{[\sum TTI)})} \nsubseteq I$.

By Lemma 2.3(v),

$$\overline{[\sum \overline{[\sum ITT)}\overline{[\sum TIT)}\overline{[\sum TTI)})} \subseteq \overline{[\sum \overline{[\sum(\sum ITT)(\sum TIT)(\sum TTI))})}$$

$$\subseteq \overline{[\sum \overline{[\sum(\sum ITTTITTTI))})}$$

$$\subseteq \overline{[\sum \overline{[\sum ITTTITTTI)})}$$

$$\subseteq \overline{[\sum \overline{[\sum ITITI)})}$$

$$\subseteq \overline{[\sum \overline{[\sum I)})}$$

$$\subseteq \overline{[\sum I)} = I.$$

Therefore $\overline{[\sum \overline{[\sum ITT)[\sum TIT)[\sum TTI)})} = I$

This is a contradiction. Therefore, $\overline{[ITT)} \subseteq I$ or $\overline{[TIT)} \subseteq I$ or $\overline{[TTI)} \subseteq I$.

Thus, Iis a prime one-sided ordered K-ideal of an ordered ternary semiring T.

Theorem 4.6. Let 'I' be an ordered K-bi-ideal of an ordered ternary semiring 'T'. Then 'I' is prime iff for a right ordered K-ideal 'P', a lateral ordered K-ideal 'Q' and a left ordered K-ideal 'R' of 'T',

$\overline{[\sum PQR)} \subseteq I \Rightarrow P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$.

Proof. Suppose that 'I' is a prime ordered K-bi-ideal of an ordered ternary semiring T.

Let P is a right ordered K-ideal, Q is a lateral ordered K-ideal and R be a left ordered K-ideal of T such that $\overline{[\sum PQR)} \subseteq I$.

Since P, Q and Rare ordered K-bi-ideals of T.

$P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$.

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Conversely, let E, F and G be any three ordered K-bi-ideals of T such that $\overline{[\sum EFG)} \subseteq I$.
Suppose that $F \nsubseteq I$ & $G \nsubseteq I$.
Let $e \in E$, $f \in F\backslash I$ and $g \in G\backslash I$.
Then $\overline{[\mathbb{N}e + eTT)} \subseteq E$, $\overline{[\mathbb{N}f + TfT)} \subseteq F$ and $\overline{[\mathbb{N}g + TTg)} \subseteq G$.
We have $\overline{[\sum \overline{[\mathbb{N}e + eTT)[\mathbb{N}f + TfT)[\mathbb{N}g + TTg)})} \subseteq \overline{[\sum EFG)} \subseteq I$.
By assumption, $\overline{[\mathbb{N}f + TfT)} \nsubseteq I$ and $\overline{[\mathbb{N}g + TTg)} \nsubseteq I$ implies that $\overline{[\mathbb{N}e + eTT)} \subseteq I$. Then $e \in I$.
Thus, $e \subseteq I$ and I is a prime ordered K-bi-ideal of an ordered ternary semiring T.

3. Fully Ordered K-bi-idempotent ternary semirings

Assume that T is an ordered ternary semiring with zero.

Definition 5.1. An ordered ternary semiring T is said to be fully ordered K-bi-idempotent if $\overline{[\sum III)} = \overline{[\sum I^3)} = I$. for any ordered ordered K-bi-ideal 'I' of T.

Theorem 5.1. Let T be an ordered ternary semiring. Then the following statements are equivalent.

(i) T is fully ordered K-bi-idempotent.
(ii) $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} = P \cap Q \cap R$. For any ordered K-bi-ideal P, Q and R of T.
(iii) Each ordered K-bi-ideal of T is semiprime.

Proof. Assume that T be an ordered ternary semiring.
(i) ⇒ (ii):
Let P, Q and R be any three ordered K-bi-ideals of an ordered ternary semiringT.
Suppose that $\overline{[\sum I^3)} = I$ for any ordered K-bi-ideal I of T. By Theorem 4.1, $P \cap Q \cap R$ an ordered K-bi-ideal of T. We have,
$P \cap Q \cap R = \overline{[\sum (P \cap Q \cap R)^3)}$
$= \overline{[\sum (P \cap Q \cap R)(P \cap Q \cap R)(P \cap Q \cap R))}$
$\subseteq \overline{[\sum PQR)}$.
Therefore, $P \cap Q \cap R \subseteq \overline{[\sum PQR)}$
Similarly, $P \cap Q \cap R \subseteq \overline{[\sum QRP)}$
and $P \cap Q \cap R \subseteq \overline{[\sum RPQ)}$.
Therefore, $P \cap Q \cap R \subseteq \overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)}$ ---------(1)
Since $\sum PQR$ is closed under addition. by Lemma 2.2, $\overline{[\sum PQR)}$ is also closed under addition.
By Lemma 2.3(vi), $(\sum PQR)(\sum PQR)(\sum PQR) \subseteq \sum PQRPQRPQR$
$\subseteq \sum (PQRPQRPQ)R$

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$\subseteq \sum\big((PQ)R(PQ)R(PQ)R\big)$

$\subseteq [\sum PQR)$

$= \sum PQR$

Therefore $(\sum PQR)(\sum PQR)(\sum PQR) \subseteq \sum PQR$, so that $\sum PQR$ is an ordered ternary subsemiring of T.

Now to prove that $\sum PQR$ is a bi-ideal of T.

By Lemma 2.3(vi), $(\sum PQR)\,T(\sum PQR)\,T(\sum PQR) \subseteq (\sum PQRT)(\sum PQRT)(\sum PQR)$

$\subseteq \sum(PQRTPQRTPQR)$

$\subseteq \sum(PQTTPQTTPQR)$

$\subseteq \sum\big((PQ)T(PQ)T(PQ)R\big)$

$\subseteq \sum\big((PQ)R\big) = \sum PQR.$

Therefore $(\sum PQR)\,T(\sum PQR)\,T(\sum PQR) \subseteq \sum PQR$.

So that $\sum PQR$ is a bi-ideal of T.

By Theorem 3.4, $\overline{[\sum PQR)}$ is an ordered K-bi-ideal of T.

Similarly, $\overline{[\sum QRP)}$ and $\overline{[\sum RPQ)}$ are an ordered K-bi-ideal.

By Theorem 3.4, $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)}$ is an ordered K-bi-ideal of T.

By assumption, Lemma 2.3(v), (vi) and Lemma 2.4, we have

$\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)}$

$= \overline{\Big[\Sigma\big(\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)}\big)\big(\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)}\big)\big(\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)}\big)\Big)}$

$\subseteq \overline{\Big[\Sigma\big(\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)}\big)\Big)}$

$\subseteq \overline{\big[\Sigma\overline{[\Sigma PQRQRPRPQ)}\big)}$

$\subseteq \overline{\big[\Sigma\overline{[\Sigma PTTTTPTTP)}\big)}$

$\subseteq \overline{\big[\Sigma\overline{[\Sigma PTPTP)}\big)}$

$\subseteq P$

Therefore $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} \subseteq P$.

Similarly, we can prove that

$\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} \subseteq Q$ and

$\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} \subseteq R$.

Thus $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} \subseteq P \cap Q \cap R$ -------------------------- (2)

Hence from equations (1) & (2)

$\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} = P \cap Q \cap R$.

(ii) ⇒ (iii):

Let I be an ordered K-bi-ideal of an ordered ternary semirings T.

Assume that $\overline{[\sum PPP)} = \overline{[\sum P^3)} \subseteq I$ for any ordered K-bi-ideal P of T.

We have $P = P \cap P \cap P$

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$= \overline{[\sum PPP)} \cap \overline{[\sum PPP)} \cap \overline{[\sum PPP)}$
$= \overline{[\sum PPP)} \subseteq I$

Thus, 'I' is semiprime.
Therefore, each ordered K-bi-ideal of T is semiprime.
(iii) ⇒ (i):
Assume that I be an ordered K-bi-ideal of T.
Since $\overline{[\sum I^3)}$ is an ordered K-bi-ideal, by assumption, $\overline{[\sum I^3)}$ is semiprime.
Since $\overline{[\sum I^3)} \subseteq \overline{[\sum I^3)}$
$I \subseteq \overline{[\sum I^3)}$, Obviously $\overline{[\sum I^3)} \subseteq I$.
Hence T is fully ordered K-bi-idempotent.

Theorem 5.2. Let T be a fully ordered K-bi-idempotent ternary semiring and 'I' be an ordered K-bi-ideal of T. Then 'I' is strongly irreducible iff 'I' is strongly prime.
Proof. Suppose that 'I' is strongly irreducible.
Let P, Q and R be any three ordered K-bi-ideals of T such that $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} \subseteq I$.
By Theorem 5.1, $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} = P \cap Q \cap R$.
Hence $P \cap Q \cap R \subseteq I$.
we have $P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$.
Therefore, 'I' is a strongly prime ordered K-bi-ideal of T.
Conversely, suppose that 'I' is strongly prime.
Let P, Q and R be any three ordered K-bi-ideals of T such that $P \cap Q \cap R \subseteq I$.
By Theorem 5.1, $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} = P \cap Q \cap R \subseteq I$.
We have $P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$.
Therefore, 'I' is a strongly irreducible ordered K-bi-ideal of T.

Theorem 5.3. Every ordered K-bi-ideal of an ordered ternary semiring T is a strongly prime ordered K-bi-ideal iff 'T' is a fully ordered K-bi-idempotent semiring and the set of all ordered K-bi-ideals of T is totally ordered.
Proof. Suppose that every ordered K-bi-ideal of T is strongly prime. Then every ordered K-bi-ideal of T is semiprime. By Theorem 5.1, T is a fully ordered K-bi-idempotent ternarysemiring.
Let P, Q and R be any three ordered K-bi-ideals of T. By Theorem 4.1, $P \cap Q \cap R$ is also an ordered
K-bi-ideal of T.
By assumption, $P \cap Q \cap R$ is a strongly prime ordered K-bi-ideal of T.
By Theorem 5.1, $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} = P \cap Q \cap R$.
Then $P \subseteq P \cap Q \cap R$or $Q \subseteq P \cap Q \cap R$ or $R \subseteq P \cap Q \cap R$.

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Thus $P = P \cap Q \cap R$ or $Q = P \cap Q \cap R$ or $R = P \cap Q \cap R$.

Therefore, $P \subseteq Q$ or $Q \subseteq R$ or $R \subseteq P$.

Conversely, suppose that T is a fully ordered K-bi-idempotent ternary semiring and the set of all ordered K-bi-ideals of T is a totally ordered set. Assume that 'I' be any ordered K-bi-ideal of T.

Let P, Q and R be any three ordered K-bi-ideals of T such that $\overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} \subseteq I$.

By Theorem 5.1, $P \cap Q \cap R = \overline{[\Sigma PQR)} \cap \overline{[\Sigma QRP)} \cap \overline{[\Sigma RPQ)} \subseteq I$.

By assumption, $P \subseteq Q$ or $Q \subseteq R$ or $R \subseteq P$.

Thus, $P \cap Q \cap R = P$ or $P \cap Q \cap R = Q$ or $P \cap Q \cap R = R$.

Therefore, $P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$.

Hence, 'I' is a strongly prime ordered K-bi-ideal of T.

Theorem 5.4. If the set of all ordered K-bi-ideals of an ordered ternary semiring T is a totally ordered set under inclusion of sets, then T is a fully ordered K-bi-idempotent iff each ordered K-bi-ideal of T is prime.

Proof. Suppose that T is a fully ordered K-bi-idempotent ternary semiring.

Let I be any ordered K-bi-ideal of T. and P, Q & R be any three ordered K-bi-ideals of T $\ni$ $\overline{[\Sigma PQR)} \subseteq I$.

we have $P \subseteq Q$ or $Q \subseteq R$ or $R \subseteq P$. Without loss of generality, Assume that $P \subseteq Q$.

Then $P = \overline{[\Sigma PPP)} \subseteq \overline{[\Sigma PPP)} \subseteq I$.

Therefore, 'I' is a prime ordered K-bi-ideal of S.

Conversely, suppose that every ordered K-bi-ideal of T is prime.

Then every ordered K-bi-ideal of T is semiprime. By Theorem 5.1, T is a fully ordered K-bi-idempotent ternary semiring.

Theorem 5.5. If T is a fully ordered k-bi-idempotent ternary semiring and 'I' is a strongly irreducible ordered K-bi-ideal of T, then 'I' is a prime ordered K-bi-ideal.

Proof. Let I be a strongly irreducible ordered K-bi-ideal of a fully ordered k-bi-idempotent ternary semiring T. Assume that P, Q and R be any three ordered K-bi-ideals of T $\ni$ $\overline{[\Sigma PQR)} \subseteq I$.

Since $P \cap Q \cap R$ is also an ordered K-bi-ideal of T.

By assumption, $\overline{[\sum (P \cap Q \cap R)^3)} = P \cap Q \cap R$

Now assume $P \cap Q \cap R = \overline{[\sum (P \cap Q \cap R)^3)}$

$= \overline{[\sum (P \cap Q \cap R)(P \cap Q \cap R)(P \cap Q \cap R))}$

$\subseteq \overline{[\Sigma PQR)} \subseteq I$.

Since 'I' is a strongly irreducible ordered K-bi-ideal of T, $P \subseteq I$ or $Q \subseteq I$ or $R \subseteq I$.

Therefore, 'I' is a prime ordered K-bi-ideal of T.

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4. Conclusion:

This research represents ordered K-bi-ideals, prime ordered K-bi-ideals, strongly prime ordered K-bi-ideals and fully ordered k-bi-idempotent in ordered ternary semirings. We have characterized a new class of ordered ternary semirings. This research further can be extended with applications of ternary semirings and ordered ternary semirings.

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